Limits and rates of change

A derivative is a limit: the average rate of change over a shorter and shorter interval settles on one number, the rate of change at an instant. This course reads that number on a table, on a graph and in a context, with its units, and finds the value a model approaches in the long term.

I THE GRADIENT AS A LIMIT

SL 5.1 · from the average rate of change to the derivative

I.1 The average rate of change

Definition 1 *average rate of change*

From $x = a$ to $x = a + h$: $\dfrac{f(a+h) - f(a)}{h}$, the gradient of the chord joining the two points of the graph. Over a time interval, an average speed.

Example 1

Find the average rate of change of $f(x) = x^2 + 1$ from $x = 1$ to $x = 4$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 The two values	$f(1) = 2,\ f(4) = 17$
Step 2 Change in f over change in x	$\dfrac{17 - 2}{4 - 1}$
Step 3 Compute	5

APPLY

1 $f(x) = x^3$, from $x = 0$ to $x = 2$.

2 A stone falls $s(t) = 5t^2$ metres in t seconds: its average speed from $t = 2$ to $t = 4$.

I.2 The limit as $h \to 0$: the derivative

Definition 2 *derivative*

As h gets closer to 0, the average rate of change gets closer to the **derivative** $f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$: the gradient of the curve at $x = a$, the **rate of change** of f at that instant; as a function, $f'(x)$ or $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

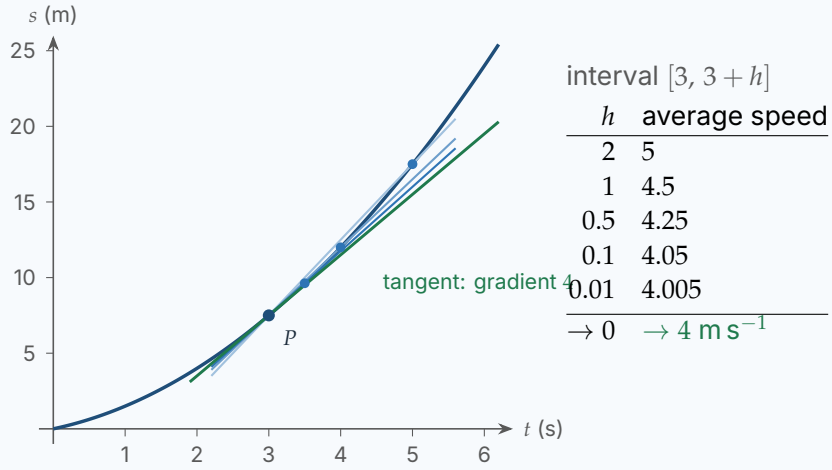
Watch out 1

h is never 0 in the fraction: simplify first, for $h \neq 0$, and only then let h approach 0.

Example 2

Let $f(x) = 3x^2$. Show that $\dfrac{f(1+h)-f(1)}{h} = 6+3h$, for $h \neq 0$, and hence find $f'(1)$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Expand $f(1+h)$, brackets kept	$3(1+h)^2 = 3+6h+3h^2$
Step 2 Subtract $f(1) = 3$	$6h+3h^2$
Step 3 Divide by h $(h \neq 0)$	$6+3h$
Step 4 Let $h \to 0$	$f'(1) = 6$



interval $[3, 3+h]$

h	average speed
2	5
1	4.5
0.5	4.25
0.1	4.05
0.01	4.005
$\to 0$	$\to 4$ m s^{-1}

Figure 1 — $s(t) = 0.5t^2 + t$ metres: the average speed over $[3, 3+h]$ is $4 + 0.5h$ and settles on 4 m s^{-1} — the same steps as Example 2.

APPLY

3 $f(x) = x^2 - 2x$: simplify $\dfrac{f(4+h)-f(4)}{h}$, then find $f'(4)$.

4 $f(x) = 4x + 1$: the rate of change at any x.

I.3 Estimating $f'(a)$ from a table

Example 3

For a function g with $g(2) = 5$, the values of $\dfrac{g(2+h) - g(2)}{h}$ are 3.9, 3.99, 4.01, 4.1 for $h = -0.1,\ -0.01,\ 0.01,\ 0.1$. Estimate $g'(2)$ and the tangent at $(2,\ 5)$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Both sides approach the same number	from below 3.99, from above 4.01: towards 4
Step 2 Estimate	$g'(2) \approx 4$
Step 3 Tangent through $(2,\ 5)$	$y - 5 = 4(x - 2),\ y = 4x - 3$

APPLY

5 The values 2.7047 $(h = -0.01)$ and 2.7319 $(h = 0.01)$: estimate the derivative to 1 decimal place.

II THE DERIVATIVE AS A RATE OF CHANGE

SL 5.1 · rates of change in context, with their units

II.1 A rate of change, with its units

Definition 3 *rate of change*

$\dfrac{\mathrm{d}V}{\mathrm{d}t} = V'(t)$ is the **rate of change of** V **with respect to** t, in units of V *per* unit of t: litres per minute, metres per second, francs per item.

Example 4

The volume of water in a tank is $V(t) = 500 - 40t + 0.8t^2$ litres after t minutes, so $V'(t) = -40 + 1.6t$. Find $V'(5)$ and interpret it.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Substitute $t = 5$	$V'(5) = -40 + 8 = -32$
Step 2 The units: litres per minute	-32 litres per minute
Step 3 Interpret: when, direction, rate	after 5 minutes the volume is decreasing at 32 litres per minute

APPLY

1 $s(t) = 3t^2 + 2t$ metres, $s'(t) = 6t + 2$: the speed at $t = 4$.

2 A profit with $P'(x) = 120 - 4x$ francs per item: interpret $P'(40)$.

II.2 The sign of the rate; a rate given

Property 1 *the sign of the rate*

$V'(t) > 0$: V is increasing at that instant; $V'(t) < 0$: decreasing. A **rate of decrease** is a positive number: decreasing at 27 per unit means $V'(t) = -27$.

Watch out 2

"Interpret" asks for a sentence: **when**, the **direction**, the **rate with its units**. "-27 litres per minute" alone earns no reasoning mark.

Example 5

For the tank of Example 4, find when the water flows out at 20 litres per minute.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Flows out at 20: a rate of decrease	$V'(t) = -20$
Step 2 Write the equation	$-40 + 1.6t = -20$
Step 3 Solve	$t = 12.5$ minutes

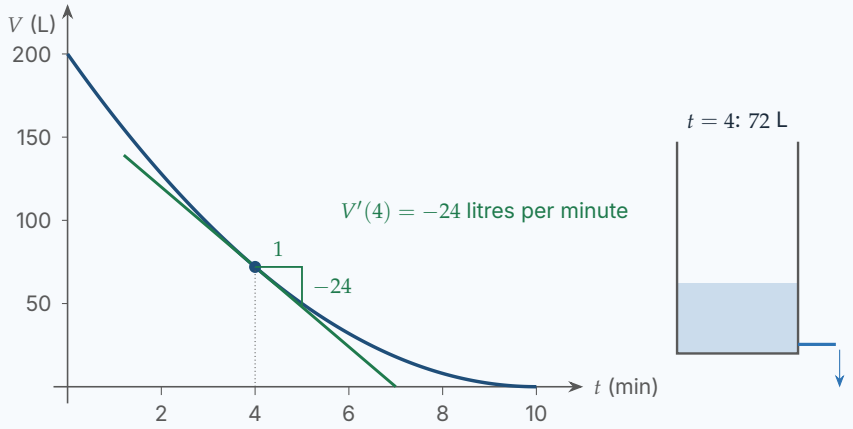


Figure 2 — $V(t) = 200 - 40t + 2t^2$ litres: at $t = 4$ the volume decreases at 24 litres per minute, the gradient of the graph.

APPLY

3 $C'(x) = 0.06x + 9$ francs per item: when does the cost increase at 15 francs per item?

4 $V'(t) = 2t - 30$: when does V decrease at 6 units per minute?

III.1 A limit at a hole

Definition 4 *limit*

$\lim_{x \to a} f(x) = L$: as x gets closer to a, from both sides, $f(x)$ gets closer to L. The value $f(a)$ plays no part, and may not exist.

Property 2 *a hole in the graph*

When $f(a)$ gives $\frac{0}{0}$, factorise, cancel the common factor (for $x \neq a$), then substitute.

Example 6

Find $\lim_{x \to 4} \dfrac{x^2 - 2x - 8}{x - 4}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Substitute: $\frac{0}{0}$	a hole at $x = 4$
Step 2 Factorise the numerator	$\dfrac{(x-4)(x+2)}{x-4}$
Step 3 Cancel, for $x \neq 4$	$x + 2$
Step 4 Substitute now	6

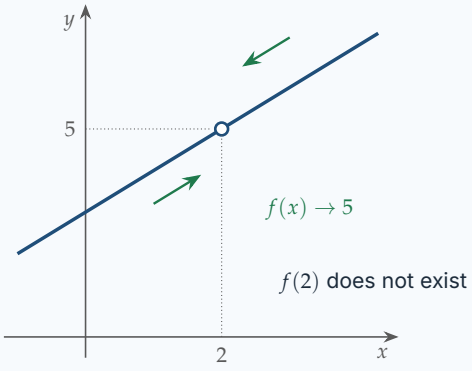


Figure 3 — $f(x) = \dfrac{x^2 + x - 6}{x - 2}$: a line with a hole at $x = 2$, and $f(x) \to 5$ from both sides — the shape of Example 6.

APPLY

1 $\lim_{x \to 1} \dfrac{x^2 + 4x - 5}{x - 1}$

2 $\lim_{x \to 1} \dfrac{2x^2 - 2}{x - 1}$

III.2 A limit estimated with values

Example 7

Let $f(x) = \dfrac{e^x - 1}{x}$, $x \neq 0$. Find $f(0.1)$, $f(0.01)$ and $f(-0.01)$, and estimate $\lim\limits_{x \to 0} f(x)$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Compute the values (GDC)	$1.05,\ 1.005,\ 0.995$
Step 2 Both sides approach the same number	towards 1
Step 3 Estimate	$\lim\limits_{x \to 0} f(x) = 1$

APPLY

3 $\dfrac{\sin x}{x}$ (x in radians) at $x = 0.01$, then the limit at 0.

III.3 The long term

Property 3 *the long term*

For $k > 0$, $e^{-kt} \to 0$ as $t \to \infty$: a model $A + B\,e^{-kt}$ approaches A, the horizontal asymptote $y = A$.

Example 8

A cup of tea has temperature $T(t) = 20 + 70\,e^{-0.05t}$ °C after t minutes. Find the starting temperature and the temperature it approaches.

WHAT YOU DO	WHAT YOU WRITE
Step 1 $t = 0$: $e^0 = 1$	$T(0) = 90\,°\text{C}$
Step 2 $e^{-0.05t} \to 0$	$T \to 20\,°\text{C}$, the temperature of the room

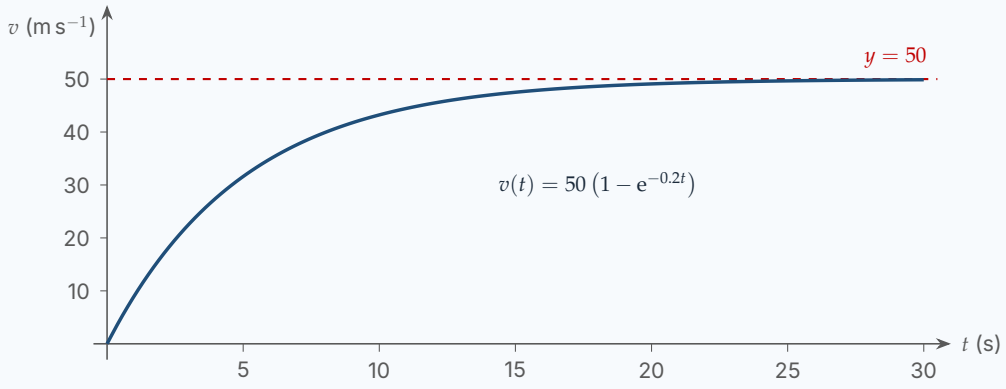


Figure 4 — $v(t) = 50\,(1 - e^{-0.2t})$ approaches 50 m s^{-1}, the asymptote $y = 50$.

APPLY

4 $P(t) = 800 - 600\,e^{-0.2t}$: $P(0)$ and the long-term value.

5 $N(t) = 50 + 30\,e^{-t}$: the value approached.

IV EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A The gradient as a limit

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 5]

Let $f(x) = x^2 + 3x$.

(a) Show that $\dfrac{f(2+h) - f(2)}{h} = 7 + h$, for $h \neq 0$. [3]

(b) Hence write down the value of $\lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$. [1]

(c) Interpret your answer to part (b) on the graph of f. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Expand $f(2+h)$, brackets kept	$f(2+h) = (2+h)^2 + 3(2+h)$	M1
	Replace x by $2+h$ everywhere, with brackets.	
	$= 10 + 7h + h^2$	A1
Step 2 Subtract $f(2)$, divide by h	$\dfrac{10 + 7h + h^2 - 10}{h} = \dfrac{7h + h^2}{h}$	A1
	$f(2) = 10$ is subtracted: the constant terms cancel, which is always the case.	
	$= 7 + h$	AG
	"Show that": the last line is the answer given, reached from the line before.	

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Let $h \to 0$	7	A1
	"Write down": no working is needed. It is $f'(2)$.	

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Say what the limit is on the graph	the gradient of the graph of f at $x = 2$ (the gradient of the tangent there)	A1
	Accept "the instantaneous rate of change of f at $x = 2$".	

EXAM-STYLE QUESTION 2

Paper 1 · no calculator · [Maximum mark: 5]

The graph of a function g passes through the point $(1, 2)$. The table shows values of $\dfrac{g(1+h)-g(1)}{h}$ for some values of h.

h	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\dfrac{g(1+h)-g(1)}{h}$	1.3393	1.3815	1.3858	1.3868	1.3911	1.4355

(a) Describe what the value 1.4355 represents on the graph of g. [1]

(b) Estimate $g'(1)$, correct to two decimal places. Justify your answer. [2]

(c) Hence write down an equation for the tangent to the graph of g at the point $(1, 2)$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE
Step 1 Say what the quotient is on the graph	the gradient of the chord joining $(1, 2)$ and the point of the graph at $x = 1.1$ **A1**
	A chord, not the tangent: $h = 0.1$ is not 0.

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE
Step 1 Both sides approach the same number	the values for $h > 0$ and for $h < 0$ approach the same number **R1**
	Both sides of the table are needed: one side alone does not justify a limit.
Step 2 Round as asked	$g'(1) \approx 1.39$ **A1**
	1.3868 and 1.3858 both round to 1.39.

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE
Step 1 Tangent through $(1, 2)$	$y - 2 = 1.39(x - 1)$ **M1**
	The point is given: only the gradient comes from (b).
Step 2 Write $y = mx + c$	$y = 1.39x + 0.61$ **A1**
	Follow through from their (b). No form is asked: $y - 2 = 1.39(x - 1)$ earns both marks.

EXAM-STYLE QUESTION 3

Paper 1 · no calculator · [Maximum mark: 7]

Water flows out of a tank. The volume of water in the tank, V litres, t minutes after the tap is opened, is given by $V(t) = 360 - 36t + 0.9t^2$, for $0 \leq t \leq 20$.

(a) Write down the volume of water in the tank when the tap is opened. [1]

(b) Find $V'(t)$. [2]

(c) Find $V'(5)$, and interpret your answer in the context of the question. [2]

(d) Find the time at which the water flows out of the tank at a rate of 18 litres per minute. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $t = 0$	360 litres	A1
	$t = 0$: the constant term.	

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Differentiate	$V'(t) = -36 + 1.8t$	A1A1
	One A1 for each term. The derivative of the constant 360 is 0.	

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Substitute $t = 5$	$V'(5) = -27$	A1
Step 2 Interpret: when, direction, rate	after 5 minutes, the volume is decreasing at 27 litres per minute	R1
	Decreasing, 27, and the units: all three are needed for R1.	

WORKING (d)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Flows out at 18: $V'(t) = -18$	$V'(t) = -18,\ -36 + 1.8t = -18$	M1
	Flowing out *at 18 litres per minute: the rate of change is -18, not 18.*	
Step 2 Solve	$t = 10$ minutes	A1
	$V'(t) = 18$ gives $t = 30$, outside the domain: a sign that the sign is wrong.	

EXAM-STYLE QUESTION 4 Paper 1 · no calculator · [Maximum mark: 6]

A workshop makes x chairs a month. The total cost of making them, in francs, is modelled by $C(x) = 0.02x^2 + 15x + 800$.

(a) Find $C'(x)$. [2]

(b) Find $C'(100)$ and interpret your answer in the context of the question. [2]

(c) Find the number of chairs for which the cost is increasing at 21 francs per chair. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Differentiate	$C'(x) = 0.04x + 15$	A1A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Substitute $x = 100$	$C'(100) = 19$	A1
Step 2 Interpret, with the units	when 100 chairs are made, the cost increases at 19 francs per additional chair	R1

The unit of $C'(x)$ is francs per chair*: the extra cost of one more chair, roughly.*

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Write the equation	$0.04x + 15 = 21$	M1
Step 2 Solve	$x = 150$ chairs	A1

C Limits, holes and the long term

EXAM-STYLE QUESTION 5 Paper 1 · no calculator · [Maximum mark: 4]

Let $f(x) = \dfrac{x^2 - 9}{x - 3}$, for $x \neq 3$.

(a) Explain why $f(3)$ cannot be found. [1]

(b) Show that $f(x) = x + 3$, for $x \neq 3$. [2]

(c) Hence write down $\lim\limits_{x \to 3} f(x)$. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Substitute: $\dfrac{0}{0}$	substituting $x = 3$ gives $\dfrac{0}{0}$: the denominator is zero	R1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Factorise the numerator	$x^2 - 9 = (x - 3)(x + 3)$ *A difference of two squares.*	M1
Step 2 Cancel, for $x \neq 3$	$\dfrac{(x - 3)(x + 3)}{x - 3} = x + 3$, for $x \neq 3$ *The cancellation is allowed because $x - 3 \neq 0$.*	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Substitute now	6	A1

EXAM-STYLE QUESTION 6

Paper 1 · no calculator · [Maximum mark: 6]

The speed of a skydiver, v metres per second, t seconds after she jumps, is modelled by $v(t) = 55\left(1 - \mathrm{e}^{-0.18t}\right)$, for $t \geq 0$.

(a) Write down the value of $v(0)$. [1]

(b) Write down the value that v approaches as t increases. [1]

(c) Show that $v'(t) = 9.9\,\mathrm{e}^{-0.18t}$. [2]

(d) Hence explain why the speed of the skydiver is increasing for all $t \geq 0$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $t = 0$	0 $\mathrm{e}^0 = 1$.	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $\mathrm{e}^{-0.18t} \to 0$	55 $\mathrm{e}^{-0.18t} \to 0$, *so* $v \to 55$*: the asymptote, never reached.*	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Write v as $55 - 55\,\mathrm{e}^{-0.18t}$	$v(t) = 55 - 55\,\mathrm{e}^{-0.18t}$, and the derivative of $\mathrm{e}^{-0.18t}$ is $-0.18\,\mathrm{e}^{-0.18t}$ *Expand first: the derivative of the constant 55 is 0.*	M1
Step 2 Differentiate	$v'(t) = -55 \times (-0.18)\,\mathrm{e}^{-0.18t}$ $= 9.9\,\mathrm{e}^{-0.18t}$	A1 AG

WORKING (d)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The sign of $v'(t)$	$\mathrm{e}^{-0.18t} > 0$ for all t, so $v'(t) > 0$ *An exponential is always positive: that is the whole argument.*	R1
Step 2 Conclude	so v is increasing *Do not award R0A1.*	A1

D With the GDC (Paper 2)

EXAM-STYLE QUESTION 7 Paper 2 · calculator · [Maximum mark: 6]

The number of bacteria in a culture, t hours after it is prepared, is modelled by $N(t) = \dfrac{2000}{1 + 19\,e^{-0.4t}}$, for $t \geq 0$.

(a) Find the number of bacteria when the culture is prepared. [1]

(b) Find the rate at which the number of bacteria is increasing when $t = 5$. [2]

(c) Write down the number of bacteria that the culture approaches in the long term. [1]

(d) Find the first time at which the number of bacteria is increasing at a rate of 100 bacteria per hour. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $t = 0$	$N(0) = 100$	A1
	$\dfrac{2000}{1 + 19}$.	

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The derivative at $t = 5$ on the GDC	use of GDC, or attempt to differentiate	(M1)
	$N'(5) = 161$ bacteria per hour $(161.281\ldots)$	A1
	Units are not required for the A1, but they cost nothing.	

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $e^{-0.4t} \to 0$	2000	A1
	$e^{-0.4t} \to 0$, *so* $N \to \dfrac{2000}{1}$.	

WORKING (d)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Write $N'(t) = 100$	$N'(t) = 100$ (seen or implied by a sketch)	(M1)
	Write the equation: on Paper 2 it carries the method mark.	
Step 2 Solve on the GDC, the first time	$t = 2.95$ hours $(2.95422\ldots)$	A1
	The rate is 100 again at $t = 11.8$: "the first time" asks for the smaller value.	

EXAM-STYLE QUESTION 8 Paper 2 · calculator · [Maximum mark: 4]

Let $f(x) = \dfrac{\sin 2x}{x}$, for $x \neq 0$, where x is in radians.

(a) Find the value of $f(0.1)$ and of $f(0.01)$. [2]

(b) Hence estimate $\lim\limits_{x \to 0} f(x)$. [1]

(c) Explain why $f(0)$ cannot be found. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Compute the values	$f(0.1) = 1.99 \ \ (1.98669\ldots)$	A1
	Radian mode: in degrees, $f(0.1) = 0.0349$.	
	$f(0.01) = 2.00 \ \ (1.99986\ldots)$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Both values approach the same number	2	A1
	The values approach 2; check with $f(-0.01)$, the same value.	

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Substitute: $\dfrac{0}{0}$	$x = 0$ gives $\dfrac{0}{0}$: the denominator is zero	R1

EXAM-STYLE QUESTION 9

Paper 2 · calculator · [Maximum mark: 6]

The depth of water in a harbour, d metres, t hours after midnight, is modelled by $d(t) = 4\cos(0.5t) + 6$, for $0 \leq t \leq 12$.

(a) Find $d'(2)$ and interpret your answer in the context of the question. [3]

(b) Find the time at which the depth of water is increasing most quickly. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The derivative at $t = 2$	use of GDC, or $d'(t) = -2\sin(0.5t)$	(M1)
	Radians.	
	$d'(2) = -1.68 \ \ (-1.68294\ldots)$	A1
Step 2 Interpret: when, direction, rate	at 2 a.m., the depth is decreasing at 1.68 metres per hour	R1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The rate is greatest: maximum of $d'(t)$	the rate of change $d'(t)$ is greatest (the maximum of the graph of d')	(M1)
	"Increasing most quickly": the largest *value of $d'(t)$, not of $d(t)$.*	
Step 2 Solve on the GDC	$t = 9.42 \ \ (3\pi)$	A1
	$\sin(0.5t) = -1$, *so* $0.5t = \frac{3\pi}{2}$.	
Step 3 Convert to a time of day	at about 9:25 a.m.	A1
	The question asks for a time: 0.425 *hours is about* 25 *minutes. Accept* 9.42 *hours after midnight.*	

V WHICH METHOD? READ THE QUESTION

5.1 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"the average rate of change of f from $x = a$ to $x = b$"	two points	$\frac{f(b)-f(a)}{b-a}$: the gradient of the chord D1
"show that $\frac{f(a+h)-f(a)}{h} = \ldots$*"*	algebra $h \neq 0$	expand $f(a+h)$ with brackets, subtract $f(a)$, divide by h D1
"hence the limit as $h \to 0$" *"the gradient at $x = a$"*	the limit of the averages	put $h = 0$ in the *simplified* expression: it is $f'(a)$ D2
"estimate $f'(a)$ from the table"	values for $h > 0$ and $h < 0$	both sides approach the same number; round as asked, and say why D2
"the rate of change of V when $t = 5$"	a derivative units	$V'(5)$, in units of V per unit of t D3
"interpret"	the sign the units	a sentence: when, increasing or decreasing, the rate with its units P1
"flows out at 18 per minute" *"decreases at a rate of ..."*	a rate of decrease	$V'(t) = -18$: the minus sign is the decrease P1
"$f(a)$ cannot be found", "find the limit"	$\frac{0}{0}$ a hole	factorise, cancel for $x \neq a$, then substitute P2
"the value approached as t increases" *"in the long term"*	$e^{-kt} \to 0$	the constant term of $A + Be^{-kt}$: the asymptote $y = A$ P3
Paper 2, a value of $f'(a)$, or a limit to estimate	the GDC is allowed	Part V: the table, the derivative at a point, a rate equal to a value

In the exam 1 *command terms*

Write down: no working needed. **Show that**: every step, the last line is the answer given. **Hence**: use the previous part. **Estimate**: a value, and the reason it is a good one. **Interpret**: a sentence in the context, with units. **Justify**: the reason earns a mark of its own (R1), and an answer without it can lose the A1 too.

VI ON THE GDC: PAPER 2

SL 5.1 · NumWorks · every calculator move of this topic

1 A table of values, to estimate a limit

Grapher → Table → Set the x values

Enter the function, then fill the column of x with values closer and closer to a, from both sides. **On paper:** copy two or three rows and write "the values approach 2".

GRAPH · TABLE

```
x f(x)
0.1 1.986693
0.01 1.999867
0.001 1.999999
```

2 The rate of change at an instant

Calculation → Toolbox → Calculus → Derivative

$f'(a)$ for any function, with no differentiation. **On paper:** "$N'(5) = 161$ (GDC)", then the units: bacteria per hour.

CALCULATION

```
diff(2000/(1+19e^(-0.4x)),
x,5)
161.2816059
```

3 When is the rate equal to a value?

Grapher → g(x) = derivative of f → Find → X given Y

Plot the derivative and the line $y =$ the rate, then Find → X given Y. **On paper:** the equation "$N'(t) = 100$", then each value of t that the question wants.

GRAPH · INTERSECTION

```
N'(x) plotted, g(x)=100
x=2.954230 y=100
x=11.76797 y=100
```

4 The long-term value

Grapher → Table → x = 20, 50, 100

Large values of t show the level the model approaches. **On paper:** the limit comes from the formula ($e^{-0.4t} \to 0$), the table only confirms it.

GRAPH · TABLE

```
x N(x)
20 1987.3
50 2000.0
100 2000.0
```

ANSWERS TO THE APPLY QUESTIONS

I.1 4

I.2 $30\ \text{m s}^{-1}$

I.3 $6+h$; 6

I.4 4

I.5 2.7

II.1 $26\ \text{m s}^{-1}$

II.2 -40: the profit decreases at 40 francs per item

II.3 $x = 100$

II.4 $t = 12$

III.1 6

III.2 4

III.3 0.99998; 1

III.4 200; 800

III.5 50

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