Limits and rates of change

Skills sheet · SL AA 5.1

KEY FACTS

Average rate from a to $a+h$: $\dfrac{f(a+h)-f(a)}{h}$ D1

Derivative: $f'(a)=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ D2

Rate of change: units of V per unit of t; $V'<0$: decreasing P1

$\mathrm{e}^{-kt}\to 0$: $A+B\,\mathrm{e}^{-kt}\to A$ P3

Simplify for $h\neq 0$ first, then let $h\to 0$ W1.

"Decreases at 5 per minute" is $V'(t)=-5$.

A hole, $\frac{0}{0}$: factorise and cancel P2.

WHAT UNLOCKS

- "show that $\dfrac{f(a+h)-f(a)}{h}=\ldots$" $\to$ brackets around $a+h$, expand, and the constants cancel.
- "estimate from the table" $\to$ look at *both* sides, $h>0$ and $h<0$: they close in on the same number.
- "interpret" $\to$ when, increasing or decreasing, how fast, in which units.
- "in the long term" $\to$ which term tends to 0? What is left is the limit.

WORKED EXAMPLE ①

Let $f(x)=x^2-4x$. Show that $\dfrac{f(3+h)-f(3)}{h}=2+h$ and hence find $f'(3)$. no GDC

$f(3+h)=(3+h)^2-4(3+h)$ M1

$=-3+2h+h^2$, $f(3)=-3$ A1

$\dfrac{2h+h^2}{h}=2+h$ A1

$f'(3)=2$ A1

Method

How do I find a limit of $\frac{f(a+h)-f(a)}{h}$?

① Write $f(a+h)$ with brackets.
② Expand; subtract $f(a)$: no constant is left.
③ Divide every term by h ($h\neq 0$).
④ Now let $h\to 0$: the terms in h vanish.

Putting $h=0$ before dividing: $\frac{0}{0}$, no mark.

WORKED EXAMPLE ②

A tank holds $V(t)=900-60t+t^2$ litres after t minutes, $0\leq t\leq 30$. Find $V'(10)$ and interpret it. no GDC

$V'(t)=-60+2t$ A1

$V'(10)=-40$ A1

after 10 minutes, the volume is decreasing at 40 litres per minute R1

Method

How do I interpret a rate?

① Find $V'(a)$, with its sign.
② The sign: increasing or decreasing.
③ The size, without the sign.
④ The units: unit of V *per* unit of t, and when.

FLUENCY

●○○ **1.** Average rate of change of $f(x) = x^3$ from $x = 1$ to $x = 3$. no GDC

●○○ **2.** $f(x) = 5x - x^2$: simplify $\dfrac{f(1+h) - f(1)}{h}$, then find $f'(1)$. no GDC

●○○ **3.** $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{x - 4}$. no GDC

●○○ **4.** $V(t) = 50 + 12t - t^2$: find $V'(3)$; is V increasing or decreasing? no GDC

●○○ **5.** $C(x) = 0.5x^2 + 20x$ francs for x items: find $C'(10)$, with its unit. no GDC

●○○ **6.** The value approached by $4 + 3\,e^{-2t}$ as t increases. no GDC

REASONING

●●○ **Question 1.** The table shows $\dfrac{f(4+h) - f(4)}{h}$ for a function f whose graph passes through $(4, 2)$. [no GDC]

h	-0.1	-0.01	0.01	0.1
value	0.25158	0.25016	0.24984	0.24846

(a) Estimate $f'(4)$. Justify your answer. [2]

(b) Hence find the equation of the tangent at $(4, 2)$, in the form $y = mx + c$. [2]

●●○ **Question 2.** The number of fish in a lake is modelled by $P(t) = 500 - 300\,e^{-0.1t}$, t in years, $t \geq 0$. [with GDC]

(a) Write down $P(0)$ and the number of fish in the long term. [2]

(b) Show that $P'(t) = 30\,e^{-0.1t}$, and explain why P is always increasing. [3]

(c) Find the rate of increase when $t = 10$. [2]

●●○ **Question 3.** The volume of a balloon, in cm^3, t seconds after it starts to be inflated, is $V(t) = 4t^2 + 2t$. [no GDC]

(a) Find the rate of change of the volume when $t = 3$. [2]

(b) Find the time at which the volume increases at 50 cm^3 per second. [2]

PROBLEMS

●●● **Question 4.** Let $f(x) = x^3$. [no GDC]

(a) Show that $\dfrac{f(a+h) - f(a)}{h} = 3a^2 + 3ah + h^2$, for $h \neq 0$. [3]

(b) Hence find $f'(a)$, and the points of the graph of f at which the gradient is 12. [3]

●●● **Question 5.** A tank empties in 40 minutes. The volume of water, in litres, is $V(t) = 1000\left(1 - \dfrac{t}{40}\right)^2$, for $0 \leq t \leq 40$. [with GDC]

(a) Show that $V(t) = 1000 - 50t + 0.625t^2$, and find $V'(t)$. [3]

(b) Find the rate at which water leaves the tank when the tank is half full. [3]

(c) Explain why the water leaves the tank fastest at $t = 0$. [2]

ANSWERS

1 $\dfrac{27-1}{3-1} = 13$

2 $f(1+h) = 4+3h-h^2$, $f(1) = 4$: $3-h$, $f'(1) = 3$

3 $\dfrac{(x-4)(x+4)}{x-4} = x+4 \to 8$

4 $V'(t) = 12-2t$, $V'(3) = 6 > 0$: increasing

5 $C'(x) = x+20$, $C'(10) = 30$ francs per item

6 4, since $\mathrm{e}^{-2t} \to 0$

Q1 (a) 0.25: the values for $h<0$ and $h>0$ approach the same number (b) $y-2 = 0.25(x-4)$: $y = 0.25x+1$

Q2 (a) 200; 500 (b) $-300\times(-0.1)\,\mathrm{e}^{-0.1t} = 30\,\mathrm{e}^{-0.1t} > 0$ (c) $P'(10) = 30\,\mathrm{e}^{-1} = 11.0$ fish per year (11.0363…)

Q3 (a) $V'(t) = 8t+2$, $V'(3) = 26\ \mathrm{cm^3\,s^{-1}}$ (b) $8t+2 = 50$: $t = 6$ s

Q4 (a) $(a+h)^3 = a^3+3a^2h+3ah^2+h^3$, subtract a^3, divide by h (b) $f'(a) = 3a^2$; $3a^2 = 12$, $a = \pm 2$: $(2,\ 8)$ and $(-2,\ -8)$

Q5 (a) $1000\left(1-\dfrac{t}{20}+\dfrac{t^2}{1600}\right)$; $V'(t) = -50+1.25t$ (b) $V = 500$ at $t = 40\left(1-\frac{1}{\sqrt{2}}\right) = 11.7$; $V' = -35.4$: water leaves at 35.4 litres per minute (c) $V'(t) = -50+1.25t$ is increasing on $[0,\ 40]$: its size $50-1.25t$ is greatest at $t = 0$

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