# Optimisation



The longest question of Paper 1 or Paper 2 is often an optimisation: a largest volume, a least cost, a greatest profit. The calculus is short; the marks are in the set-up (one variable, its domain), the justification, and the answer to the question actually asked. Each part ends with exam-style questions, marked line by line.

## I OPTIMISING A GIVEN FUNCTION

SL 5.8 · the greatest or least value of a model

### I.1 The greatest value of a model

**Property 1** *the method*

**(1)** solve $Q'(x) = 0$; **(2)** justify the maximum or minimum, by $Q'$ or $Q''$; **(3)** on a closed domain, compare with the ends; **(4)** answer the question, with units.

**Watch out 1**

"Find the maximum profit" asks for a value of $P$, not of $x$: read the question again before the last line.

**Example 1**

A profit, in thousands of euros, is $P(x) = -2x^2 + 48x - 100$. Find the value of $x$ that gives the greatest profit, and that profit.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Solve $P'(x) = 0$ | $-4x + 48 = 0$: $x = 12$ |
| **Step 2** Justify | $P''(x) = -4 < 0$: a maximum |
| **Step 3** Answer: the value of $P$ | $P(12) = 188$ thousand euros |

**APPLY**

1 $C(x) = 2x^2 - 20x + 70$: the least cost.

2 $h(t) = 30t - 5t^2$ metres: the greatest height.

## I.2 A closed domain: the ends count

**Example 2**

Find the greatest and the least value of $f(x) = x^3 - 6x^2 + 9x$ for $0 \le x \le 5$.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Stationary points | $3(x-1)(x-3) = 0$: $f(1) = 4, f(3) = 0$ |
| **Step 2** The ends | $f(0) = 0,\ f(5) = 20$ |
| **Step 3** Compare | greatest 20 (at $x = 5$), least 0 |

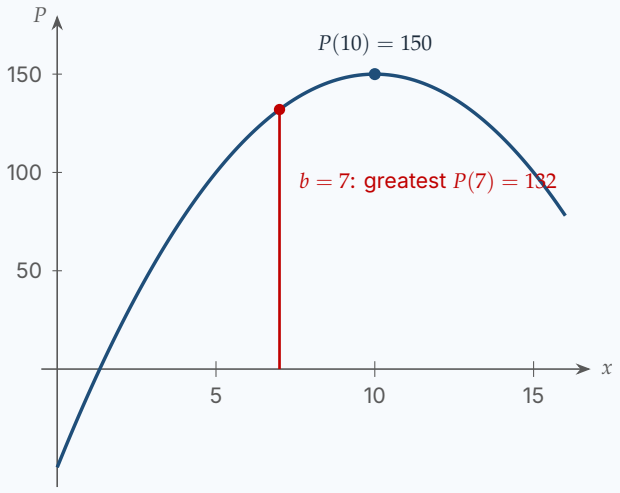



*Figure 1 — $P(x) = -2x^2 + 40x - 50$ on $0 \le x \le b$: the greatest value is at $x = 10$ while it is inside the domain, at the end $x = b$ otherwise.*

**APPLY**

**3** $R(p) = p(300 - 10p)$, with $p \ge 18$: the greatest revenue.

**4** $f(x) = x^2 - 6x$, $0 \le x \le 4$: greatest and least.

# II SETTING UP THE FUNCTION

SL 5.8 · one variable, from a constraint

## II.1 From two variables to one

**Property 2** *from two variables to one*

A **constraint** (fixed perimeter, area, volume) links the two variables: solve it for one, substitute, and write the domain of the other.

**Example 3**

Positive numbers $x$ and $y$ satisfy $x + 2y = 30$. Find the greatest value of $A = xy$.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Solve the constraint for $y$ | $y = \dfrac{30 - x}{2}$ |
| **Step 2** Substitute: one variable | $A = 15x - \dfrac{x^2}{2}$ |
| **Step 3** Solve $A'(x) = 0$, justify | $15 - x = 0$: $x = 15$; $A'' = -1 < 0$ |
| **Step 4** Answer | $A = 15 \times 7.5 = 112.5$ |

**APPLY**

1 A rectangle has perimeter 36 cm: its greatest area.

2 $x + y = 10$: the least value of $x^2 + y^2$.

## II.2 A box, or a can

**Watch out 2**

"Show that $V = \ldots$": write the constraint on the page, then the substitution.

**Example 4**

An open box is made from a 24 cm square sheet by cutting a square of side $x$ cm from each corner. Find the greatest volume.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Volume with one variable | $V = x(24 - 2x)^2$, $0 < x < 12$ |
| **Step 2** Solve $V'(x) = 0$ | $(24 - 2x)(24 - 6x) = 0$: $x = 4$ ($x = 12$ rejected) |
| **Step 3** Compute | $V = 4 \times 16^2 = 1024$ cm$^3$ |

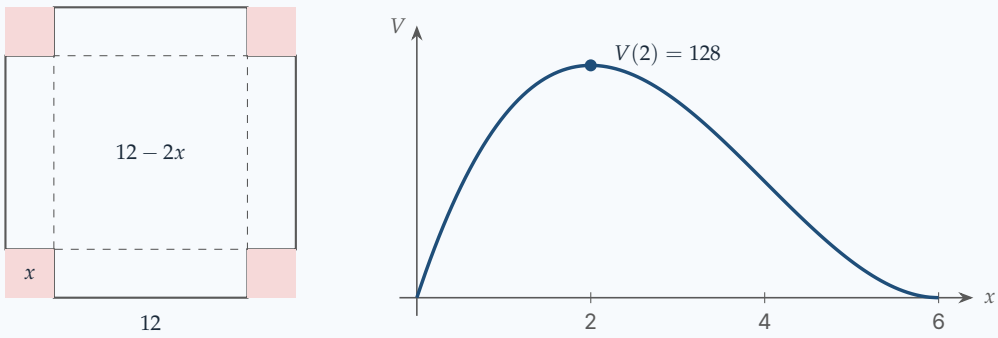



*Figure 2 — From a 12 cm sheet, $V(x) = x(12 - 2x)^2$ is greatest at $x = 2$ — the same steps as Example 4.*

**APPLY**

**3** A closed box with a square base of side $x$ and volume 1000 cm$^3$: the least surface area.

**4** An open can of volume $64\pi$ cm$^3$: the least area of metal.

## III.1 Three ways to justify

**Property 3** *three ways to justify*

The **second derivative** at the point; the **sign of the first derivative** on each side; on Paper 2, the **graph**, sketched. On a closed interval, compare the ends too.

**Example 5**

A closed can has volume $54\pi$ cm$^3$, so its surface area is $S = 2\pi r^2 + \dfrac{108\pi}{r}$. Find the least surface area, and justify.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Solve $S'(r) = 0$ | $4\pi r - \dfrac{108\pi}{r^2} = 0$: $r^3 = 27$, $r = 3$ |
| **Step 2** Justify with $S''$ | $S'' = 4\pi + \dfrac{216\pi}{r^3} > 0$: a minimum |
| **Step 3** Answer | $S = 18\pi + 36\pi = 54\pi$ cm$^2$ |

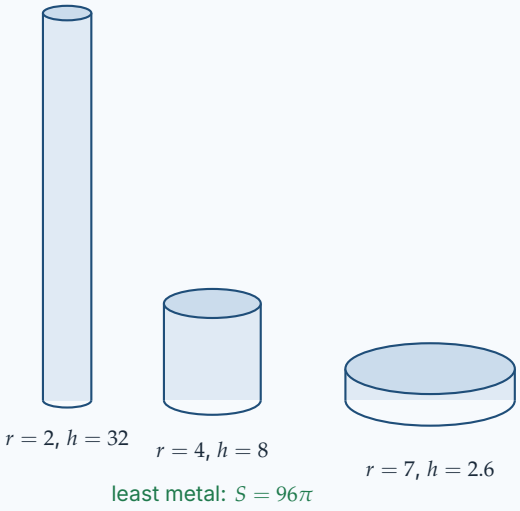



*Figure 3 — Cans of volume $128\pi$: the least metal at $r = 4$, $h = 8$ — height equal to diameter, as for Example 5's can ($r = 3$, $h = 6$).*

**APPLY**

**1** $A(x) = 60x - 3x^2$: justify the maximum at $x = 10$ with the sign of $A'$.

## III.2 A root outside the domain; whole numbers

**Watch out 3**

A root of $Q'(x) = 0$ outside the domain is rejected, with the reason. For whole numbers of items, compare the two integers either side.

**Example 6**

A daily profit is $P(n) = -3n^2 + 100n - 200$ euros for $n$ items, $n$ a whole number. Find the best $n$ and the greatest profit.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Solve $P'(n) = 0$ | $n = 16.67$, not a whole number |
| **Step 2** Compare the integers either side | $P(16) = 632,\ P(17) = 633$ |
| **Step 3** Answer | 17 items, 633 euros |

**APPLY**

**2** $P(n) = -n^2 + 25n$, $n$ whole: the greatest value.

## III.3 On the GDC

**Example 7**

Find the point $P\left(x, e^x\right)$ of the curve $y = e^x$ nearest to $A(0, 2)$.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** The distance with one variable | $AP = \sqrt{x^2 + (e^x - 2)^2}$ |
| **Step 2** Minimum of its graph on the GDC | $x = 0.524,\ AP = 0.609$ |
| **Step 3** The point | $P(0.524, 1.69)$ |

**APPLY**

**3** Cost $C(x) = 3(8 - x) + 5\sqrt{x^2 + 4}$, $0 \le x \le 8$: the least cost.

# IV EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

## A A given function

**EXAM-STYLE QUESTION 1** Paper 1 · no calculator · [Maximum mark: 6]

A company makes and sells $x$ thousand units of a product a month. Its monthly profit, in thousands of euros, is modelled by $P(x) = -0.4x^2 + 24x - 200$, for $0 \leq x \leq 50$.

(a) Find $P'(x)$. [1]

(b) Find the value of $x$ that gives the greatest profit. Justify your answer. [3]

(c) Find the greatest monthly profit. [2]

**WORKING (a)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Differentiate | $P'(x) = -0.8x + 24$ | A1 |

**WORKING (b)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Solve $P'(x) = 0$ | $-0.8x + 24 = 0$ | M1 |
| | $x = 30$ | A1 |
| **Step 2** Justify with $P''$ | $P''(x) = -0.8 < 0$, so a maximum | R1 |
| | *Or: $P'$ changes from $+$ to $-$ at $x = 30$.* | |

**WORKING (c)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Substitute | $P(30) = -360 + 720 - 200$ | (M1) |
| | *Substitute into $P$, not into $P'$.* | |
| **Step 2** Answer with units | 160 thousand euros | A1 |
| | *Accept 160, or 160 000 euros.* | |

**EXAM-STYLE QUESTION 2** Paper 1 · no calculator · [Maximum mark: 6]

A theatre sells $600 - 20x$ tickets for a show when the price of a ticket is $x$ francs. The revenue from the tickets is $R(x) = x(600 - 20x)$ francs.

(a) Find the price that gives the greatest revenue, and this revenue. [4]

The theatre decides that the price of a ticket must be at least 18 francs.

(b) Find the greatest revenue under this condition. Justify your answer. [2]

**WORKING (a)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Differentiate | $R'(x) = 600 - 40x$<br>*Expand first: $R(x) = 600x - 20x^2$.* | A1 |
| **Step 2** Solve $R'(x) = 0$ | $600 - 40x = 0$ | M1 |
| | $x = 15$ francs | A1 |
| **Step 3** The revenue | $R(15) = 4500$ francs | A1 |

**WORKING (b)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** $R$ decreases after 15: the end counts | $R'(x) < 0$ for $x > 15$: $R$ decreases on $x \geq 18$, so the greatest value is at $x = 18$<br>*The stationary point is outside the allowed prices: the end of the domain wins.* | R1 |
| **Step 2** $R(18)$ | $R(18) = 18 \times 240 = 4320$ francs | A1 |

## B Setting up the function

**EXAM-STYLE QUESTION 3** Paper 1 · no calculator · [Maximum mark: 7]

An open box is made from a rectangular sheet of card 30 cm by 16 cm, by cutting a square of side $x$ cm from each corner and folding up the sides.

(a) Show that the volume of the box is $V = x(30 - 2x)(16 - 2x)$, and write down the possible values of $x$. [2]

(b) Show that $\dfrac{\mathrm{d}V}{\mathrm{d}x} = 4(3x - 10)(x - 12)$. [3]

(c) Find the value of $x$ that gives the greatest volume, and this volume. [2]

**WORKING (a)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Base and height from the folding | base $(30 - 2x)$ by $(16 - 2x)$, height $x$ | R1 |
| | *Each side of the base loses $x$ at both ends.* | |
| **Step 2** Every length positive | $0 < x < 8$ | A1 |
| | *$16 - 2x > 0$: the narrow side decides.* | |

**WORKING (b)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Expand | $V = 4x^3 - 92x^2 + 480x$ | A1 |
| | *Expand before differentiating.* | |
| **Step 2** Differentiate | $\dfrac{\mathrm{d}V}{\mathrm{d}x} = 12x^2 - 184x + 480$ | A1 |
| **Step 3** Factorise | $= 4\left(3x^2 - 46x + 120\right) = 4(3x - 10)(x - 12)$ | A1 |

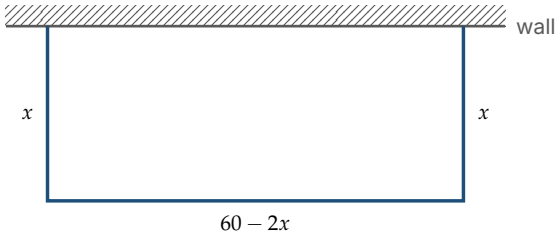
**WORKING (c)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Reject the root outside the domain | $x = \dfrac{10}{3}$ $(x = 12$ is outside $0 < x < 8)$ | A1 |
| | *The other root is rejected, and the reason written.* | |
| **Step 2** Compute | $V = \dfrac{10}{3} \times \dfrac{70}{3} \times \dfrac{28}{3} = \dfrac{19600}{27}$ cm$^3$ | A1 |
| | *$\approx 726$ cm$^3$.* | |

## EXAM-STYLE QUESTION 4

Paper 1 · no calculator · [Maximum mark: 7]

A farmer has 60 m of fencing to make a rectangular enclosure against a long straight wall. The wall forms one side, so only three sides need fencing. The sides perpendicular to the wall have length $x$ metres.




(a) Show that the area of the enclosure is $A = 60x - 2x^2$. [2]

(b) Find the value of $x$ for which the area is greatest. Justify your answer. [3]

(c) Find the greatest area. [2]

### WORKING (a)

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** The third side from the fencing | the third side is $60 - 2x$ | M1 |
| | *The constraint:* $2x + y = 60$. | |
| **Step 2** The area | $A = x(60 - 2x) = 60x - 2x^2$ | A1 |

### WORKING (b)

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Solve $A'(x) = 0$ | $A'(x) = 60 - 4x = 0$ | M1 |
| | $x = 15$ | A1 |
| **Step 2** Justify with $A''$ | $A''(x) = -4 < 0$: a maximum | R1 |

### WORKING (c)

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Substitute | $A = 15 \times 30$ | (M1) |
| **Step 2** Answer with units | $450\ \text{m}^2$ | A1 |

**EXAM-STYLE QUESTION 5** Paper 1 · no calculator · [Maximum mark: 6]

A closed cylindrical can has radius $r$ cm and height $h$ cm. Its volume is $128\pi$ cm$^3$.

(a) Show that the total surface area of the can is $S = 2\pi r^2 + \dfrac{256\pi}{r}$. [2]

(b) Find the value of $r$ for which $S$ is least. Justify your answer. [3]

(c) Find the least surface area, as a multiple of $\pi$. [1]

---

**WORKING (a)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Solve the volume for $h$ | $\pi r^2 h = 128\pi$, so $h = \dfrac{128}{r^2}$<br>*The constraint, solved for h.* | M1 |
| **Step 2** Surface area | $S = 2\pi r^2 + 2\pi r h$<br>*Two discs and the curved surface.* | A1 |
| **Step 3** Substitute | $= 2\pi r^2 + 2\pi r \times \dfrac{128}{r^2} = 2\pi r^2 + \dfrac{256\pi}{r}$ | AG |

**WORKING (b)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Solve $S'(r) = 0$ | $\dfrac{\mathrm{d}S}{\mathrm{d}r} = 4\pi r - \dfrac{256\pi}{r^2} = 0$ | M1 |
| | $r^3 = 64,\ r = 4$ | A1 |
| **Step 2** Justify with $S''$ | $\dfrac{\mathrm{d}^2S}{\mathrm{d}r^2} = 4\pi + \dfrac{512\pi}{r^3} > 0$: a minimum | R1 |

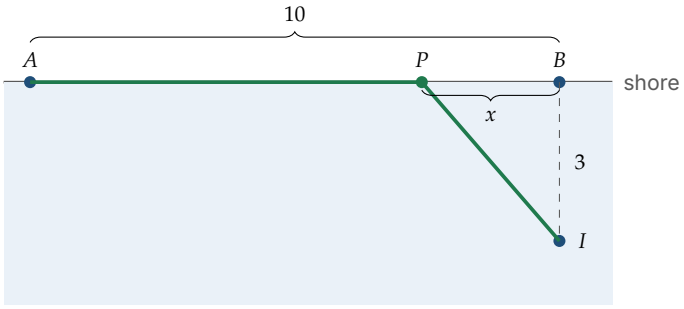
**WORKING (c)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Substitute $r = 4$ | $S = 32\pi + 64\pi = 96\pi$ cm$^2$ | A1 |

## C With the GDC (Paper 2)

**EXAM-STYLE QUESTION 6** Paper 2 · calculator · [Maximum mark: 6]

A cable joins a station $A$ on a straight shore to an island $I$, 3 km from the shore. The point $B$ of the shore nearest to $I$ is 10 km from $A$. The cable runs along the shore from $A$ to a point $P$, then under water from $P$ to $I$. Let $BP = x$ km. Cable costs 2.4 million euros per km along the shore and 4.1 million euros per km under water.




(a) Show that the cost of the cable, in millions of euros, is $C(x) = 2.4(10 - x) + 4.1\sqrt{x^2 + 9}$, for $0 \leq x \leq 10$. [2]

(b) Find the value of $x$ that gives the least cost, and this least cost. [4]

**WORKING (a)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Along the shore | $AP = 10 - x$ | A1 |
| **Step 2** Under water: Pythagoras | $PI = \sqrt{x^2 + 9}$, by Pythagoras | R1 |

**WORKING (b)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Minimum on the GDC | use of GDC (minimum of the graph), or $C'(x) = 0$ | (M1) |
| **Step 2** Read $x$ | $x = 2.17$ km $(2.16596\ldots)$ | A1 |
| **Step 3** Read $C$ | $C = 34.0$ million euros $(33.9724\ldots)$ | A1 |
| **Step 4** Compare with the ends | less than $C(0) = 36.3$ and $C(10) = 42.8$ | R1 |

*The ends of the interval are compared: the minimum is global.*

## EXAM-STYLE QUESTION 7

Paper 2 · calculator · [Maximum mark: 5]

Let $A$ be the point $(1, 3)$, and $P(x, \ln x)$ a point on the curve $y = \ln x$, for $x > 0$.

(a) Write down an expression for the distance $AP$ in terms of $x$. [1]

(b) Find the coordinates of the point $P$ for which the distance $AP$ is least, and this least distance. [4]

### WORKING (a)

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Distance formula | $AP = \sqrt{(x-1)^2 + (\ln x - 3)^2}$ | A1 |

### WORKING (b)

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Minimum on the GDC | minimum of the graph of $AP$ on the GDC<br>*The minimum of $AP^2$ is at the same $x$: easier, if done by hand.* | (M1) |
| **Step 2** Read $x$ | $x = 2.09 \quad (2.08582\ldots)$ | A1 |
| **Step 3** The point | $P(2.09, 0.735)$<br>*$y = \ln x$: the point is on the curve.* | A1 |
| **Step 4** The distance | $AP = 2.51 \quad (2.51167\ldots)$ | A1 |

# V WHICH METHOD? READ THE QUESTION

5.8 · the whole topic on one page

| THE QUESTION SAYS | YOU KNOW | YOU WRITE |
|---|---|---|
| *"find the maximum / minimum value of $Q$"* | one variable | $Q'(x) = 0$, justify, then $Q$ at that $x$, with units P1 |
| *"show that $V = \ldots$", two variables* | a constraint | write the constraint, solve it for one variable, substitute P2 |
| *"write down the possible values of $x$"* | lengths are positive | every length positive: the domain, e.g. $0 < x < 8$ P2 |
| *"justify"* | three ways | $Q'' < 0$ or $> 0$; the sign of $Q'$ either side; a sketch on Paper 2 P3 |
| *a closed interval, or a condition on $x$* | the ends count | compare the stationary value with the values at the ends P1 |
| *a root outside the domain* | rejected | say why: "$x = 12$ is not possible since $x < 8$" W3 |
| *whole numbers of items* | integers | compare the two integers either side of the stationary point W3 |
| *Paper 2, a function too long to differentiate* | the GDC is allowed | Part V: Maximum, Minimum on the graph, with the domain as window |

**In the exam 1** *command terms*

**Show that**: the constraint and the substitution written. **Find the value of** $x$: the variable; **find the maximum**: the quantity. **Justify**: a reason that earns its own mark. **Exact**: keep $\pi$, fractions, roots. Units: in the answer, always.

# VI ON THE GDC: PAPER 2

SL 5.8 · NumWorks · every calculator move of this topic

## 1 The minimum of a quantity

Grapher → Axes → the domain → Minimum

Window = the domain, so that the minimum shown is one the question allows. **On paper:** "$C'(x) = 0$ or the minimum on the GDC", then $x$ and $C$, to three significant figures.

**GRAPHER · MINIMUM**

```
f(x)=2.4(10-x)
+4.1*sqrt(x^2+9)
X: 0 to 10
x=2.165965 y=33.97246
```

### 2 Compare with the ends

Calculation → f(0) → f(10)

On a closed interval, the greatest or least value may be at an end. Two lines on the GDC, written on the page: the justification of a global answer.

**CALCULATION**

```
f(0)
36.3
f(10)
42.80525669
```

### 3 A distance to a curve

Grapher → f(x) = distance → Find → Minimum

The distance from $(a, b)$ to the point $\big(x, g(x)\big)$ is $\sqrt{(x-a)^2+\big(g(x)-b\big)^2}$: enter it as a function of $x$ and find its minimum.

**GRAPHER · MINIMUM**

```
f(x)=sqrt((x-1)^2
+(ln(x)-3)^2)
x=2.085824 y=2.511672
```

## ANSWERS TO THE APPLY QUESTIONS

**I.1** 20, at $x = 5$

**I.2** 45 m, at $t = 3$

**I.3** $R(18) = 2160$ ($R$ decreases after $p = 15$)

**I.4** 0; $-9$

**II.1** 81 cm$^2$ (a square)

**II.2** 50

**II.3** $S = 2x^2 + \frac{4000}{x}$: 600 cm$^2$

**II.4** $r = 4$: $48\pi$ cm$^2$

**III.1** $A'(9) = 6 > 0$, $A'(11) = -6 < 0$

**III.2** 156, at $n = 12$ or 13

**III.3** 32, at $x = 1.5$

Written and verified by a practising IB mathematics teacher, produced with AI assistance; every result recomputed by software, every figure computed from its function. Not affiliated with or endorsed by the International Baccalaureate Organization. © 2026 Hence