Stationary points and inflexion

Where $f'(x) = 0$ the tangent is horizontal: the graph turns, or only pauses. The exam asks three things about these points: where they are, what they are, and why. This course gives the two tests and the reasons that earn the R1, then the questions where the coefficients, or the number of solutions, are unknown.

I STATIONARY POINTS AND THEIR NATURE

SL 5.8 · $f'(x) = 0$, and the first derivative test

I.1 Finding the stationary points

Definition 1 *stationary point*

A **stationary point**: $f'(x) = 0$, the tangent is horizontal. A **local maximum** if $f(a)$ is the greatest value near a, a **local minimum** if it is the least.

Watch out 1

"Coordinates" means both: x from $f'(x) = 0$, then $y = f(x)$, from the function, not from f'.

Example 1

Find the coordinates of the stationary point of $f(x) = x^2 - 6x + 11$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Solve $f'(x) = 0$	$2x - 6 = 0$: $x = 3$
Step 2 y from f	$f(3) = 2$
Step 3 Write the point	$(3, 2)$

APPLY

1 $y = 5 + 4x - x^2$

2 $y = x^3 - 48x$

I.2 The first derivative test

Property 1 *the first derivative test*

At $x = a$ with $f'(a) = 0$: f' from $+$ to $-$, a local maximum; from $-$ to $+$, a local minimum; same sign on both sides, a **stationary point of inflexion**.

Example 2

Find the stationary points of $f(x) = 2x^3 - 9x^2 + 12x$ and determine their nature with the sign of f'.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Solve $f'(x) = 0$, factorised	$6(x-1)(x-2) = 0$: $(1, 5)$, $(2, 4)$
Step 2 Sign of f' on each side	$f'(0) = 12$, $f'(1.5) = -1.5$, $f'(3) = 12$
Step 3 Conclude	$(1, 5)$ local maximum, $(2, 4)$ local minimum

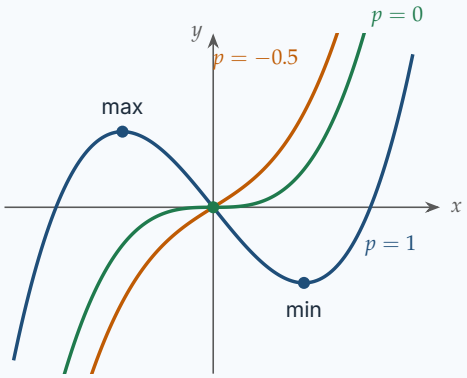


Figure 1 — $f(x) = x^3 - 3px$: two stationary points at $x = \pm\sqrt{p}$ when $p > 0$, one stationary point of inflexion when $p = 0$, none when $p < 0$.

APPLY

3 $y = x + \dfrac{16}{x}$, $x > 0$: the stationary point and its nature.

4 $y = x^3 + 3x^2 + 3x$: the stationary point and its nature.

II THE SECOND DERIVATIVE TEST AND POINTS OF INFLEXION

SL 5.8 · $f''(a)$, and inflexions with or without a zero gradient

II.1 The second derivative test

Property 2 *the second derivative test*

$f'(a) = 0$ and $f''(a) < 0$: a **local maximum**; $f''(a) > 0$: a **local minimum**; $f''(a) = 0$: the test says *nothing*, go back to the sign of f'.

Example 3

Let $f(x) = x^3 + 3x^2 - 24x$. Find its stationary points and use the second derivative to determine their nature.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Solve $f'(x) = 0$	$3(x+4)(x-2) = 0$: $(-4, 80)$, $(2, -28)$
Step 2 $f''(x)$	$6x + 6$
Step 3 Sign of f'' at each point	$f''(-4) = -18 < 0$: maximum; $f''(2) = 18 > 0$: minimum

APPLY

1 $y = x\,\mathrm{e}^{-x}$: the stationary point and its nature.

2 $y = x^2 + \dfrac{54}{x}$: the stationary point and its nature.

II.2 When $f''(a) = 0$: two kinds of inflexion

Property 3 *two kinds of inflexion*

At a point of inflexion, f'' changes sign. If also $f'(a) = 0$: a **stationary point of inflexion**; otherwise a **non-stationary point of inflexion**.

Watch out 2

$f''(a) = 0$ does not mean an inflexion (x^4 has a minimum at 0), nor "neither": the test has failed. Use the sign of f' on each side.

Example 4

Let $f(x) = x^4 + 4x^3$. Find the stationary points and their nature.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Solve $f'(x) = 0$	$4x^2(x+3) = 0$: $x = 0, x = -3$
Step 2 Second derivative at each	$f''(x) = 12x^2 + 24x$: $f''(-3) = 36 > 0$, $f''(0) = 0$
Step 3 $x = -3$	local minimum $(-3, -27)$
Step 4 $x = 0$: the sign of f'	$f'(-1) = 8 > 0$, $f'(1) = 16 > 0$: stationary inflexion $(0, 0)$

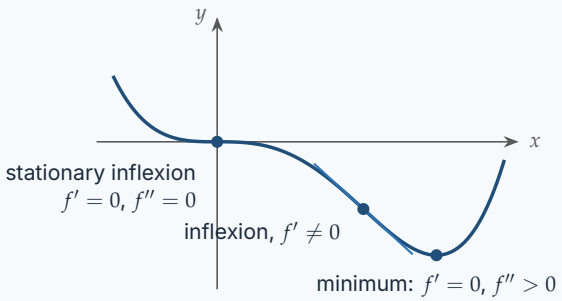


Figure 2 — $f(x) = 3x^4 - 8x^3$: a stationary inflexion at 0, a non-stationary inflexion at $\frac{4}{3}$, a minimum at 2.

APPLY

3 $y = (x-1)^3 + 2$: the stationary point and its nature.

4 $y = x^4 + 2$: the nature of $(0, 2)$.

III UNKNOWN COEFFICIENTS AND THE NUMBER OF SOLUTIONS

SL 5.8 · SL 2.10 · working back from a stationary point

III.1 A stationary point given

Property 4 *a stationary point given*

"A stationary point at $(a,\ b)$" gives **two** equations: $f'(a) = 0$ and $f(a) = b$.

Example 5

The graph of $f(x) = ax^2 + bx$ has a stationary point at $(2,\ -8)$. Find a and b.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Stationary: $f'(2) = 0$	$4a + b = 0$
Step 2 On the graph: $f(2) = -8$	$4a + 2b = -8$
Step 3 Solve	$b = -8,\ a = 2$

APPLY

1 $f(x) = x^3 + px + q$ has a stationary point at $(1,\ 5)$. Find p and q.

2 $f(x) = ax^3 - 12x$ has a stationary point at $x = 2$. Find a.

III.2 How many solutions has $f(x) = k$?

Property 5 *the number of solutions of $f(x) = k$*

Where $y = k$ meets the graph. A cubic with local maximum value M and minimum value m: three solutions when $m < k < M$, two when $k = m$ or $k = M$, one otherwise.

Example 6

Let $f(x) = x^3 - 12x$. Find the values of k for which $f(x) = k$ has three solutions.

WHAT YOU DO	WHAT YOU WRITE
Step 1 The turning points	maximum $(-2,\ 16)$, minimum $(2,\ -16)$
Step 2 Strictly between the two values	$-16 < k < 16$

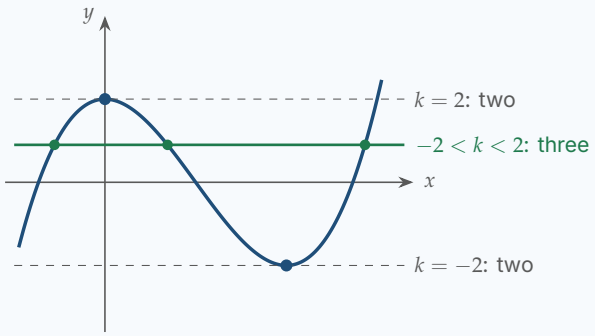


Figure 3 — $y = x^3 - 3x^2 + 2$: the line $y = k$ meets it three times exactly when $-2 < k < 2$.

APPLY

3 $f(x) = x^3 - 3x^2$: k for three solutions of $f(x) = k$.

4 Same f: how many solutions has $f(x) = 1$?

III.3 Greatest and least values on an interval

Example 7

Find the greatest and the least value of $f(x) = x^3 - 12x$ for $-1 \leq x \leq 5$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Stationary points inside the interval	$x = 2$: $f(2) = -16$
Step 2 The values at the ends	$f(-1) = 11$, $f(5) = 65$
Step 3 Compare	greatest 65, least -16

APPLY

5 $f(x) = x^2 - 4x + 1$, $0 \leq x \leq 5$: greatest and least values.

6 $f(x) = 4x - x^2$, $0 \leq x \leq 3$: greatest value.

III.4 On the GDC

Example 8

Let $f(x) = x^2 \, e^{-0.5x}$, for $0 \leq x \leq 10$. Find the local maximum point.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Graph f on the GDC	one maximum inside the domain
Step 2 Maximum	$x = 4$, $y = 2.165\ldots$
Step 3 Write the point, 3 s.f.	$(4, 2.17)$

APPLY

7 $f(x) = x \ln x$, $x > 0$: the local minimum point.

IV EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A Stationary points and their nature

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 6]

Let $f(x) = x^3 - 3x^2 - 24x + 5$.

(a) Find the coordinates of the stationary points of the graph of f. [4]

(b) Determine the nature of each stationary point. Justify your answer. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Solve $f'(x) = 0$	$f'(x) = 3x^2 - 6x - 24 = 0$	M1
Step 2 Factorise	$3(x-4)(x+2) = 0$: $x = 4,\ x = -2$	A1
Step 3 y from f	$(4,\ -75)$	A1
	$f(4) = 64 - 48 - 96 + 5$.	
	$(-2,\ 33)$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Sign of f' on each side	f' changes from $+$ to $-$ at $x = -2$, and from $-$ to $+$ at $x = 4$ (or $f''(-2) = -18 < 0$, $f''(4) = 18 > 0$)	R1
	Either test, with its numbers or its signs.	
Step 2 Conclude	$(-2,\ 33)$ is a local maximum, $(4,\ -75)$ a local minimum	A1

EXAM-STYLE QUESTION 2

Paper 1 · no calculator · [Maximum mark: 5]

Let $f(x) = x + \frac{4}{x}$, for $x > 0$.

(a) Show that the graph of f has a stationary point at $(2, 4)$. [3]

(b) Use the sign of $f'(x)$ to determine the nature of this stationary point. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Differentiate	$f'(x) = 1 - 4x^{-2} = 1 - \frac{4}{x^2}$	A1
Step 2 $f'(2) = 0$	$f'(2) = 1 - \frac{4}{4} = 0$ *"Show that" a stationary point: substitute, and show the zero.*	M1
Step 3 $f(2) = 4$	$f(2) = 2 + 2 = 4$ *The y-coordinate is part of what is to be shown.*	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Sign of f' on each side	$f'(1) = -3 < 0$ and $f'(3) = \frac{5}{9} > 0$: f' changes from $-$ to $+$ *Any values either side of 2 in the domain $x > 0$.*	R1
Step 2 Conclude	local minimum *Do not award R0A1.*	A1

B The second derivative test

EXAM-STYLE QUESTION 3

Paper 1 · no calculator · [Maximum mark: 6]

Let $f(x) = x^3 - 12x + 7$.

(a) Find the coordinates of the stationary points of the graph of f. [4]

(b) Use the second derivative to determine the nature of each stationary point. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Solve $f'(x) = 0$	$f'(x) = 3x^2 - 12 = 0$	M1
	$x = \pm 2$	A1
Step 2 y from f	$(2, -9)$	A1
	$(-2, 23)$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Sign of f'' at each point	$f''(x) = 6x$: $f''(2) = 12 > 0$, $f''(-2) = -12 < 0$	M1
	The sign of f'' at each point, written.	
Step 2 Conclude	minimum at $(2, -9)$, maximum at $(-2, 23)$	A1

EXAM-STYLE QUESTION 4

Paper 1 · no calculator · [Maximum mark: 7]

Let $f(x) = 3x^4 + 4x^3 + 1$.

(a) Find the x-coordinates of the stationary points of the graph of f. [3]

(b) Use the second derivative to show that the graph of f has a local minimum at $x = -1$. [2]

(c) Show that the point $(0, 1)$ is a point of inflexion of the graph of f. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Differentiate	$f'(x) = 12x^3 + 12x^2$	A1
Step 2 Factorise	$12x^2(x + 1) = 0$	M1
Step 3 Solve	$x = 0$, $x = -1$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $f''(-1)$	$f''(x) = 36x^2 + 24x$, $f''(-1) = 12$	A1
Step 2 Conclude	$f''(-1) > 0$, so a local minimum	R1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Sign of f'' on each side of 0	$f''(x) = 12x(3x + 2)$: $f'' < 0$ for $-\frac{2}{3} < x < 0$, $f'' > 0$ for $x > 0$	M1
	$f''(0) = 0$ is not enough: the change of sign is the proof.	
Step 2 Conclude	f'' changes sign at $x = 0$, so $(0, 1)$ is a point of inflexion	R1
	Here $f'(0) = 0$ too: a stationary point of inflexion. $f(0) = 1$.	

C Working back, and f(x) = k

EXAM-STYLE QUESTION 5

Paper 1 · no calculator · [Maximum mark: 6]

Let $f(x) = x^3 + px^2 + q$, where p and q are constants. The graph of f has a stationary point at $(2, -1)$.

(a) Find the value of p and the value of q. [4]

(b) Find the coordinates of the other stationary point, and determine its nature. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Stationary: $f'(2) = 0$	$f'(x) = 3x^2 + 2px,\ f'(2) = 12 + 4p = 0$ *Stationary: the derivative is zero at $x = 2$.*	M1
Step 2 Solve for p	$p = -3$	A1
Step 3 On the graph: $f(2) = -1$	$f(2) = 8 - 12 + q = -1$ *On the graph: the coordinates satisfy $y = f(x)$.*	M1
Step 4 Solve for q	$q = 3$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The other zero of f'	$3x^2 - 6x = 0$: $x = 0$, the point $(0, 3)$	A1
Step 2 Nature: $f''(0)$	$f''(0) = -6 < 0$: a local maximum *$f''(x) = 6x - 6$.*	A1

EXAM-STYLE QUESTION 6

Paper 1 · no calculator · [Maximum mark: 6]

Let $f(x) = 2x^3 + 3x^2 - 36x$.

(a) Find the coordinates of the local maximum point and of the local minimum point of the graph of f. [4]

(b) Find the values of k for which the equation $f(x) = k$ has three solutions. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Solve $f'(x) = 0$	$f'(x) = 6x^2 + 6x - 36 = 6(x + 3)(x - 2) = 0$	M1
	$x = -3,\ x = 2$	A1
Step 2 The maximum	maximum $(-3, 81)$ *A cubic with a positive leading coefficient: the maximum comes first.*	A1
Step 3 The minimum	minimum $(2, -44)$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Between the two values	k between the minimum value and the maximum value *A sketch with a horizontal line earns this mark.*	(M1)
Step 2 Write the interval	$-44 < k < 81$ *Strict: $k = 81$ gives two solutions only.*	A1

D With the GDC (Paper 2)

EXAM-STYLE QUESTION 7 Paper 2 · calculator · [Maximum mark: 6]

Let $f(x) = \sin 2x + \cos x$, for $0 \leq x \leq \pi$, where x is in radians.

(a) Find the coordinates of the local maximum point and of the local minimum point of the graph of f. [4]

(b) Find the coordinates of the point of inflexion of the graph of f. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Maximum and Minimum on the GDC	use of GDC (maximum and minimum of the graph)	(M1)
	maximum $(0.635, 1.76)$ $x = 0.634866\ldots, y = 1.76017\ldots$	A1
	minimum $(2.51, -1.76)$ $x = 2.50672\ldots$	A1
	both labelled correctly	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Turning point of f'	a turning point of f', or $f''(x) = 0$	(M1)
Step 2 The point	$(1.57, 0)$ $\left(\frac{\pi}{2}, 0\right)$ *The other zeros of f'' are outside $0 \leq x \leq \pi$.*	A1

EXAM-STYLE QUESTION 8

Paper 2 · calculator · [Maximum mark: 6]

Let $g(x) = \ln\left(x^2 + 1\right) - 0.3x$, for $0 \leq x \leq 8$.

(a) Find the x-coordinates of the stationary points of the graph of g. [2]

(b) Find the greatest value of $g(x)$ for $0 \leq x \leq 8$. [2]

(c) Find the least value of $g(x)$ for $0 \leq x \leq 8$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Zeros of g' on the GDC	$x = 0.154 \ \ (0.153535\ldots)$	A1
	$\frac{2x}{x^2+1} = 0.3$; *exact:* $x = \frac{10 \pm \sqrt{91}}{3}$.	
	$x = 6.51 \ \ (6.51313\ldots)$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Compare with the ends	compare $g(6.51) = 1.817$ with the ends $g(0) = 0$ and $g(8) = 1.774$	M1
	The ends of the interval are candidates too.	
Step 2 The greatest value	$1.82 \ \ (1.81700\ldots)$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Compare with the ends	compare $g(0.154) = -0.0228$ with $g(0) = 0$ and $g(8)$	M1
Step 2 The least value	$-0.0228 \ \ (-0.0227610\ldots)$	A1
	A small negative number: the graph dips just below the axis before it rises.	

V WHICH METHOD? READ THE QUESTION

5.8 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"find the stationary points", "the coordinates"	$f'(x) = 0$	solve $f'(x) = 0$; then $y = f(x)$ for each x D1
"determine the nature", "justify"	two tests	$f''(a)$ and its sign, or the signs of f' either side; the reason before the word P1
"use the second derivative"	$f''(a)$	$f''(a) < 0$: maximum; > 0: minimum; $= 0$: no conclusion, use f' P2
"point of inflexion"	f'' changes sign	solve $f''(x) = 0$, show the change of sign; if also $f' = 0$, it is stationary P3
"a stationary point at (a, b)", unknown constants	two conditions	$f'(a) = 0$ and $f(a) = b$: two equations P4
"$f(x) = k$ has three solutions"	the max and min values	$y = k$ strictly between the minimum value and the maximum value P5
"greatest value on $a \leq x \leq b$"	an interval	compare the local maxima and the values at the ends
Paper 2, coordinates to find	the GDC is allowed	Part V: Maximum, Minimum, the ends of the domain

In the exam 1 *command terms*

Find the coordinates: x and y. **Determine the nature, justify**: a test with its result written, then the word maximum or minimum (R1, then A1). **Show that**: the zero of f', and the value of f, both written. **Use the second derivative**: the first derivative test then earns nothing.

VI ON THE GDC: PAPER 2

SL 5.8 · NumWorks · every calculator move of this topic

1 Local maximum and minimum

Grapher → OK on the curve → Find → Maximum

Set the window to the domain first: a turning point outside it is not an answer. **On paper:** the coordinates, to three significant figures, and the word maximum or minimum.

GRAPHER · MAXIMUM

```
f(x)=sin(2x)+cos(x)
x=0.6348669 y=1.760173
Find → Minimum: x=2.506726
y=-1.760173
```

2 A point of inflexion

Grapher → g(x) = derivative of f → Find → Minimum

At an inflexion, f' has a turning point: find it on the graph of $g = f'$, then evaluate f there. **On paper:** the coordinates of the point of f.

GRAPHER · MINIMUM

```
g(x)=f'(x)
x=1.570796 y=-3
f(1.570796)=0
```

3 The greatest value on an interval

Grapher → Axes → x from 0 to 8 → Maximum

The greatest value is a local maximum *or* a value at an end of the interval: compute both ends. **On paper:** the three values compared, then the answer.

GRAPHER · MAXIMUM

```
f(x)=ln(x^2+1)-0.3x
x=6.513131 y=1.817001
f(8)=1.774387 f(0)=0
```

ANSWERS TO THE APPLY QUESTIONS

I.1 $(2,\ 9)$

I.2 $(4,\ -128),\ (-4,\ 128)$

I.3 $(4,\ 8)$, minimum

I.4 $(-1,\ -1)$, stationary inflexion

II.1 $\left(1,\ e^{-1}\right)$, maximum

II.2 $(3,\ 27)$, minimum

II.3 $(1,\ 2)$, stationary inflexion

II.4 minimum ($f''(0) = 0$, but f' goes from $-$ to $+$)

III.1 $p = -3, q = 7$

III.2 $a = 1$

III.3 $-4 < k < 0$

III.4 one

III.5 $6;\ -3$

III.6 4

III.7 $(0.368,\ -0.368)$

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