The graphs of f, f′ and f″

Three graphs tell one story. The sign of f' says where f rises and falls; its zeros are the turning points; the sign of f'' says which way the graph bends. Exam questions give one graph or one formula and ask about another: this course reads each graph from the next.

I INCREASING AND DECREASING

SL 5.2 · the sign of f'

I.1 The sign of f'

Definition 1 *increasing, decreasing*

f is **increasing** where $f'(x) > 0$: the graph rises from left to right; **decreasing** where $f'(x) < 0$. Where $f'(x) = 0$, the tangent is horizontal.

Example 1

Let $f(x) = x^2 - 8x + 3$. Find where f is decreasing and where it is increasing.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Differentiate	$f'(x) = 2x - 8$
Step 2 Solve $f'(x) < 0$	$x < 4$: decreasing
Step 3 And $f'(x) > 0$	$x > 4$: increasing

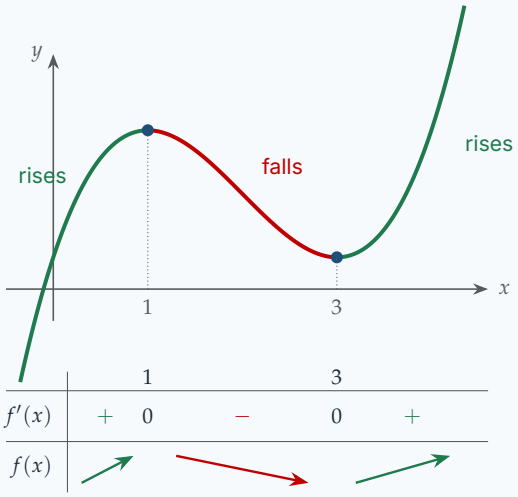


Figure 1 — $f(x) = x^3 - 6x^2 + 9x + 1$: the graph rises where $f'(x) = 3(x-1)(x-3)$ is positive, falls where it is negative.

APPLY

1 $f(x) = 5 - 3x$: where is f decreasing?

2 $f(x) = x^3 + 2x$: where is f increasing?

I.2 A sign table

Property 1 *a sign table*

Factorise $f'(x)$, find its zeros, give the sign of f' on each interval between them. The answer is an interval of x.

Watch out 1

The interval is a set of values of x, never of y, with strict inequalities: $-2 < x < 1$.

Example 2

Let $f(x) = x^3 - 3x^2 - 24x$. Find the values of x for which f is decreasing.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Differentiate, factorise	$f'(x) = 3x^2 - 6x - 24 = 3(x-4)(x+2)$
Step 2 Zeros of f'	$x = -2$ and $x = 4$
Step 3 Sign between the zeros	$f'(0) = -24 < 0$
Step 4 Write the interval	$-2 < x < 4$

APPLY

3 $f(x) = 2x^3 - 9x^2 + 12x$: where is f increasing?

4 $f(x) = x^3 - 27x$: where is f decreasing?

I.3 A factor that is always positive

Example 3

Let $f(x) = x^2\,\mathrm{e}^x$. Find where f is decreasing.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Product rule, factorise	$f'(x) = (x^2 + 2x)\,\mathrm{e}^x = x(x+2)\,\mathrm{e}^x$
Step 2 $\mathrm{e}^x > 0$: the other factors decide	sign of $x(x+2)$
Step 3 Write the interval	$-2 < x < 0$

APPLY

5 $f(x) = (x+3)\,\mathrm{e}^{-x}$: where is f increasing?

II FROM ONE GRAPH TO THE OTHER

SL 5.2 · SL 5.7 · the graph of f' read on f, and f read on f'

II.1 Reading f' on the graph of f

Property 2 *reading f' on the graph of f*

Where f rises, f' is above the axis; where f falls, below. The turning points of f are the zeros of f'; the steepest points of f are the maximum and minimum of f'.

Example 4

The graph of f has a local maximum at $(-1, 4)$ and a local minimum at $(2, -5)$, and no other turning point. Write down $f'(-1)$, where $f'(x) < 0$, and the sign of $f'(3)$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Turning point: $f' = 0$	$f'(-1) = 0$
Step 2 Falling between the turning points	$-1 < x < 2$
Step 3 Rising after the minimum	$f'(3) > 0$

APPLY

1 f has only a minimum, at $(3, 1)$: where is $f'(x) > 0$?

2 Same f: the sign of $f'(0)$.

II.2 Reading f on the graph of f'

Property 3 *reading f on the graph of f'*

f has a **local maximum** where f' changes from $+$ to $-$, a **local minimum** where it changes from $-$ to $+$. Where the graph of f' only *touches* the axis, f has a stationary point that is neither.

Watch out 2

A maximum of the graph of f' is not a maximum of f: it is where f rises fastest, a point of inflexion of f.

Example 5

The graph of f' crosses the x-axis from positive to negative at $x = -3$, from negative to positive at $x = 1$, and has its minimum at $x = -1$. Describe the graph of f at these three values.

WHAT YOU DO	WHAT YOU WRITE
Step 1 $+$ to $-$	$x = -3$: local maximum of f
Step 2 $-$ to $+$	$x = 1$: local minimum of f
Step 3 A turning point of f'	$x = -1$: point of inflexion of f

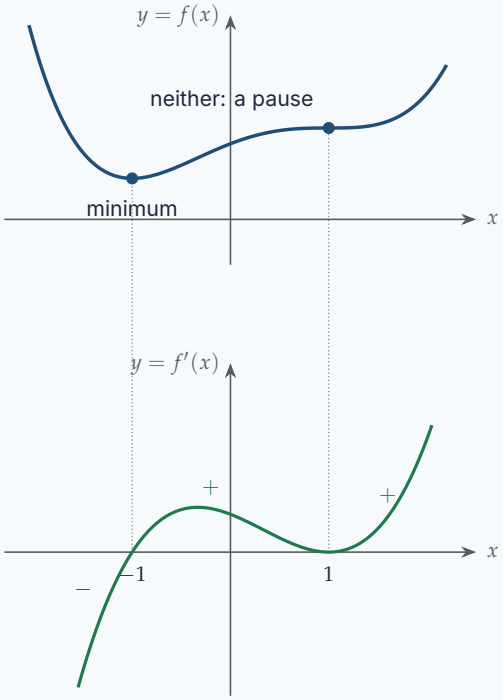


Figure 2 — $f'(x) = 0.5(x+1)(x-1)^2$ and the graph of f: a minimum at $x = -1$; at $x = 1$, f' touches zero and f only pauses.

APPLY

3 $f'(x) = (x-1)^2(x+4)$: the nature of each stationary point of f.

III CONCAVITY AND THE SECOND DERIVATIVE

SL 5.7 · the sign of f'', points of inflexion

III.1 Concavity: the sign of f''

Definition 2 *second derivative*

The **second derivative** $f''(x)$, or $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$, is the derivative of $f'(x)$: the rate of change of the gradient.

Definition 3 *concavity*

$f''(x) > 0$: the graph is **concave up** (a cup, above its tangents). $f''(x) < 0$: **concave down** (below its tangents).

Example 6

Let $f(x) = x^3 - 9x^2 + 2$. Find where the graph of f is concave up.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Differentiate twice	$f'(x) = 3x^2 - 18x,\ f''(x) = 6x - 18$
Step 2 Solve $f''(x) > 0$	$x > 3$

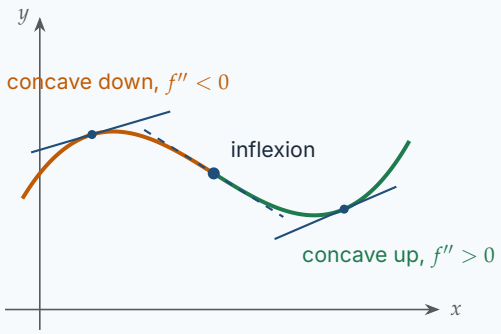


Figure 3 — Concave down, the curve lies below its tangent; concave up, above. At the point of inflexion the tangent crosses the curve.

APPLY

1 $f(x) = x^4 - 6x^2$: where is it concave down?

2 $f(x) = \mathrm{e}^x + x^2$: where is it concave up?

III.2 Points of inflexion

Property 4 *point of inflexion*

A **point of inflexion**: the concavity changes, $f''(x)$ **changes sign**. Solve $f''(x) = 0$, then check the change of sign; f' has a maximum or minimum there.

Watch out 3

$f''(a) = 0$ is not enough: write the reason, "f'' changes sign".

Example 7

Let $f(x) = x^3 + 6x^2 - 5$. Find the point of inflexion of its graph.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Solve $f''(x) = 0$	$6x + 12 = 0$: $x = -2$
Step 2 Check the change of sign	$f'' < 0$ for $x < -2$, > 0 for $x > -2$
Step 3 y from f	$(-2,\ 11)$

APPLY

3 $f(x) = 2x^3 - 3x^2$: the point of inflexion.

4 $f(x) = x^4 + 1$: a point of inflexion at $x = 0$?

III.3 On the GDC

Example 8

Let $f(x) = \ln\left(x^2 + 1\right) - 0.8x$. Find where f is decreasing, and the point of inflexion with $x > 0$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Graph f'; its zeros	$x = 0.5$ and $x = 2$
Step 2 f decreasing where $f' < 0$	$x < 0.5$ or $x > 2$
Step 3 Inflexion: the maximum of f'	$x = 1$: $(1,\ -0.107)$

APPLY

5 $g'(x) = 2\cos x - x$, $0 \le x \le 2$: where is g increasing?

IV EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A Increasing and decreasing

EXAM-STYLE QUESTION 1

Paper 1 · no calculator · [Maximum mark: 5]

Let $f(x) = 2x^3 + 3x^2 - 12x + 4$.

(a) Find $f'(x)$. [2]

(b) Find the values of x for which f is decreasing. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Differentiate	$f'(x) = 6x^2 + 6x - 12$	A2
	A1 for two correct terms.	

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Solve $f'(x) < 0$, factorised	$6x^2 + 6x - 12 < 0,\ 6(x+2)(x-1) < 0$	M1
	Decreasing: $f'(x) < 0$. Factorise to find the zeros.	
Step 2 Zeros of f'	zeros $x = -2$ and $x = 1$	A1
Step 3 Write the interval	$-2 < x < 1$	A1
	Between the zeros: f' is a parabola opening upwards, negative between its roots.	

EXAM-STYLE QUESTION 2

Paper 1 · no calculator · [Maximum mark: 4]

Let $f(x) = x\,\mathrm{e}^{-x}$.

(a) Show that $f'(x) = (1 - x)\,\mathrm{e}^{-x}$. [2]

(b) Find the values of x for which f is increasing. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Product rule	product rule: $f'(x) = 1 \times \mathrm{e}^{-x} + x \times (-\mathrm{e}^{-x})$	M1A1
	M1 for the product rule, A1 for $(\mathrm{e}^{-x})' = -\mathrm{e}^{-x}$.	
	$= (1 - x)\,\mathrm{e}^{-x}$	AG

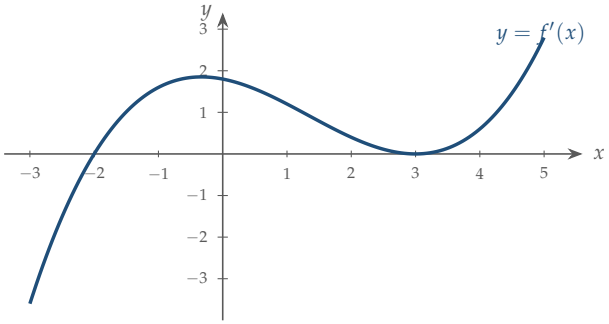
WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $\mathrm{e}^{-x} > 0$: the other factor decides	$\mathrm{e}^{-x} > 0$, so $f'(x)$ has the sign of $1 - x$	R1
	The exponential is always positive: only $1 - x$ decides the sign.	
Step 2 Write the interval	$x < 1$	A1
	Do not award R0A1.	

B From one graph to the other

EXAM-STYLE QUESTION 3 Paper 1 · no calculator · [Maximum mark: 7]

The diagram shows the graph of $y = f'(x)$, the derivative of a function f, for $-3 \leq x \leq 5$. The graph of f' crosses the x-axis at $x = -2$, has a local maximum at $x = -\frac{1}{3}$, and touches the x-axis at $x = 3$.



(a) Write down the x-coordinates of the stationary points of the graph of f. [2]
(b) Determine the nature of the stationary point at $x = -2$. Justify your answer. [2]
(c) Explain why the graph of f has neither a maximum nor a minimum at $x = 3$. [1]
(d) Write down the x-coordinates of the points of inflexion of the graph of f. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Zeros of f'	$x = -2$	A1
	Stationary points of f: the zeros of f'.	
	$x = 3$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The change of sign	f' changes from negative to positive at $x = -2$	R1
Step 2 Name it	local minimum	A1
	Do not award R0A1.	

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 f' keeps its sign	f' does not change sign at $x = 3$: it is positive on both sides	R1
	$f'(3) = 0$ alone is not a reason: it says the point is stationary, not what kind.	

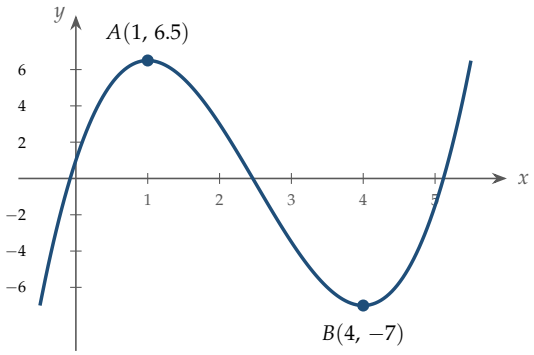
WORKING (d)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Turning points of f'	$x = -\frac{1}{3}$	A1
	Where f' has a turning point: here its maximum.	
	$x = 3$	A1
	The minimum of f': a stationary point of inflexion of f.	

EXAM-STYLE QUESTION 4

Paper 1 · no calculator · [Maximum mark: 5]

The diagram shows the graph of a cubic function f, for $-0.5 \leq x \leq 5.5$. The graph has a local maximum at $A(1, 6.5)$ and a local minimum at $B(4, -7)$.



(a) Write down the value of $f'(1)$. [1]

(b) Write down the values of x for which $f'(x) < 0$. [1]

(c) Write down whether $f'(5)$ is positive or negative. [1]

(d) The graph of f' is a parabola. Find the x-coordinate of the point at which the graph of f is decreasing most steeply. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Turning point: $f' = 0$	0	A1
	A turning point: the tangent is horizontal.	

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Where f falls	$1 < x < 4$	A1
	Where the graph of f falls.	

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Where f rises	positive	A1
	f rises after B.	

WORKING (d)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Steepest point: the vertex of f'	the minimum of f', at the vertex of the parabola, halfway between its zeros 1 and 4	R1
	A parabola is symmetric about the vertical line through its vertex.	
Step 2 Halfway between 1 and 4	$x = 2.5$	A1

C Concavity and points of inflexion

EXAM-STYLE QUESTION 5 Paper 1 · no calculator · [Maximum mark: 6]

Let $f(x) = x^3 - 6x^2 + 5x + 2$.

(a) Find $f''(x)$. [2]

(b) Find the values of x for which the graph of f is concave up. [2]

(c) Hence find the coordinates of the point of inflexion of the graph of f. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Differentiate	$f'(x) = 3x^2 - 12x + 5$	(A1)
Step 2 Differentiate again	$f''(x) = 6x - 12$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Solve $f''(x) > 0$	$6x - 12 > 0$ *Concave up: $f''(x) > 0$.*	M1
Step 2 Write the interval	$x > 2$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 f'' changes sign at $x = 2$	f'' changes sign at $x = 2$ *From (b): concave down for $x < 2$, up for $x > 2$.*	R1
Step 2 y from f	$(2, -4)$ *$f(2) = 8 - 24 + 10 + 2 = -4$, a value of f.*	A1

EXAM-STYLE QUESTION 6

Paper 1 · no calculator · [Maximum mark: 6]

Let $f(x) = x^4 - 4x^3$.

(a) Find $f''(x)$. [2]

(b) Find the x-coordinates of the points of inflexion of the graph of f. Justify your answer. [3]

(c) Write down the coordinates of these points. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Differentiate	$f'(x) = 4x^3 - 12x^2$	(A1)
Step 2 Differentiate again	$f''(x) = 12x^2 - 24x$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Solve $f''(x) = 0$	$12x(x-2) = 0$: $x = 0,\ x = 2$	M1
Step 2 Check the changes of sign	f'' changes sign at each: $+$, then $-$ on $0 < x < 2$, then $+$	R1
	Without the change of sign, $x = 0$ and $x = 2$ are only candidates.	
Step 3 Conclude	$x = 0$ and $x = 2$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 y from f	$(0,\ 0)$ and $(2,\ -16)$	A1

D With the GDC (Paper 2)

EXAM-STYLE QUESTION 7 Paper 2 · calculator · [Maximum mark: 6]

Let $f(x) = e^{0.5x} - x^2$, for $0 \leq x \leq 6$.

(a) Find $f'(x)$. [2]

(b) Find the values of x for which f is decreasing. [2]

(c) Find the coordinates of the point of inflexion of the graph of f. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Differentiate	$f'(x) = 0.5\,e^{0.5x} - 2x$	A1A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Zero of f' on the GDC	$f'(x) < 0$; zero of f' at $x = 0.289$ $(0.288842\ldots)$ *A sketch of f': negative from its zero to the end of the domain.*	(M1)
Step 2 Write the interval	$0.289 < x \leq 6$ *Accept $0.289 < x < 6$. The other zero of f', 6.52, is outside the domain.*	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $f'' = 0$, or the minimum of f'	$f''(x) = 0.25\,e^{0.5x} - 2 = 0$, or the minimum of the graph of f': $x = 4.16$ *Exact: $x = 2\ln 8$.*	(M1)
Step 2 The point	$(4.16, -9.30)$ *$y = f(4.15888\ldots) = -9.2963\ldots$: a value of f.*	A1

EXAM-STYLE QUESTION 8

Paper 2 · calculator · [Maximum mark: 6]

The derivative of a function g is given by $g'(x) = 3\sin x - x$, for $0 \leq x \leq 3$, where x is in radians.

(a) Find the values of x for which g is increasing. [2]

(b) Find $g''(x)$. [1]

(c) Find the x-coordinate of the point of inflexion of the graph of g. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Zeros of g' on the GDC	$g'(x) > 0$; zeros of g' at $x = 0$ and $x = 2.28$ $(2.27886\ldots)$ *Plot g' itself: here it is the function given.*	(M1)
Step 2 Write the interval	$0 < x < 2.28$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Differentiate	$g''(x) = 3\cos x - 1$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Solve $g''(x) = 0$	$3\cos x - 1 = 0$	M1
Step 2 Compute	$x = 1.23$ $(1.23095\ldots)$ *Exact: $x = \arccos\frac{1}{3}$.*	A1
Step 3 Check the change of sign	g'' changes sign at $x = 1.23$ (from positive to negative) *A sketch of g'', or the maximum of g' there.*	R1

V WHICH METHOD? READ THE QUESTION

5.7 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"find where f is increasing / decreasing"	the sign of f'	factorise $f'(x)$; a sign table; an interval of x P1
$f'(x) = (\ldots)\,\mathrm{e}^{\cdots}$	$\mathrm{e}^{\cdots} > 0$	the sign is decided by the other factor alone; say so D1
the graph of f is given, "about f'"	rises: $f' > 0$ turning point: $f' = 0$	read the direction of f; its steepest point is the max or min of f' P2
the graph of f' is given, "about f"	sign change $+$ to $-$ $-$ to $+$ touches	maximum, minimum, neither; a turning point of f' is an inflexion of f P3
"justify", "determine the nature"	a reason	"f' changes from … to …", written before the answer (R1, then A1) P3
"concave up / down"	the sign of f''	$f''(x) > 0$ or < 0, an interval of x D3
"point of inflexion"	$f'' = 0$ and a sign change	solve $f''(x) = 0$, check the change of sign, then $y = f(x)$ P4
Paper 2, the graph of f' on the GDC	the GDC is allowed	Part V: intervals from the zeros of f', inflexions from the turning points of f'

In the exam 1 *command terms*

Write down: read on the graph, no working needed. **Determine**, **justify**: the reason earns R1, and the answer without it earns nothing. **Hence**: use the previous part. An **interval** is a set of values of x, with strict inequalities where f' or f'' is zero at the ends.

VI ON THE GDC: PAPER 2

SL 5.2 · 5.7 · NumWorks · every calculator move of this topic

1 The graph of f', to read where f increases

Grapher → g(x) = derivative of f → Find → Zeros

Plot f' as g; its zeros cut the domain into intervals, and the graph shows the sign on each. **On paper:** "$f'(x) < 0$", then the interval.

GRAPHER · ZEROS

```
g(x)=diff(f(x),x,x)
x=0.2888427 y=0
g > 0 for 0 < x < 0.289
```

2 A point of inflexion

Grapher → g(x) = derivative of f → Find → Minimum

At a point of inflexion, f' has a turning point: find the maximum or minimum of g. Then the y-coordinate is f at that x, not g. **On paper:** the coordinates, to three significant figures.

GRAPHER · MINIMUM

```
g(x)=f'(x)
x=4.158883 y=-4.317766
f(4.158883)=-9.296309
```

3 Where does f' change sign?

Grapher → f(x) = derivative given → Find → Zeros

When the question gives $f'(x)$ itself, plot it as the function. Its zeros and its sign answer "increasing" and "maximum"; its turning points answer "inflexion".

GRAPHER · ZEROS

```
f(x)=3sin(x)-x
x=0 y=0
x=2.278863 y=0
```

ANSWERS TO THE APPLY QUESTIONS

I.1 everywhere: $f'(x) = -3 < 0$

I.2 everywhere: $3x^2 + 2 > 0$

I.3 $x < 1$ or $x > 2$

I.4 $-3 < x < 3$

I.5 $f'(x) = -(x+2)\mathrm{e}^{-x}$: $x < -2$

II.1 $x > 3$

II.2 negative

II.3 $x = -4$ minimum; $x = 1$ neither

III.1 $-1 < x < 1$

III.2 everywhere: $f''(x) = \mathrm{e}^x + 2 > 0$

III.3 $\left(\frac{1}{2}, -\frac{1}{2}\right)$

III.4 no: $f''(x) = 12x^2$ does not change sign

III.5 $0 \leq x < 1.03$

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