Exponential and logarithmic functions

SL AA · Functions · 2.9 · 2.10

An exponential multiplies by the same factor at each step: it grows, or decays, faster than any polynomial, and never reaches its asymptote. The logarithm undoes it; its graph is the same curve, mirrored in the line $y = x$.

I EXPONENTIAL FUNCTIONS

SL 2.9 · a^x, e^x, growth and decay, the asymptote

I.1 The graph of a^x

Definition 1 *exponential function*

For $a > 0$, $a \neq 1$: $f(x) = a^x$, domain $\mathbb{R}$, range $y > 0$. The graph passes through $(0, 1)$, with the horizontal asymptote $y = 0$. The number $e \approx 2.718$ gives e^x.

Example 1

Let $f(x) = 3^x$. Find $f(2)$, $f(0)$ and $f(-1)$, then write down the asymptote and the range.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Substitute A negative power is a fraction, never negative.	$f(2) = 9,\ f(0) = 1,\ f(-1) = \frac{1}{3}$
Step 2 Asymptote: the value never reached	$y = 0$
Step 3 Range: above the asymptote	$f(x) > 0$

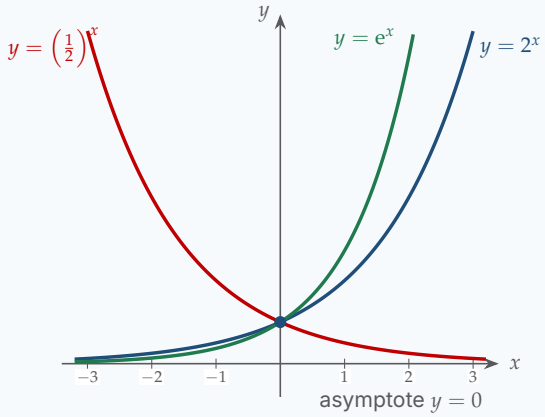


Figure 1 — $y = 2^x$, $y = e^x$ and $y = \left(\frac{1}{2}\right)^x$: all pass through $(0, 1)$, as 3^x does in Example 1, and stay above $y = 0$.

APPLY

1 $f(x) = 5^x$: find $f(-2)$.

2 $f(x) = \left(\frac{1}{2}\right)^x$: find $f(-3)$.

3 Range of e^x.

I.2 Growth or decay

Property 1 *growth and decay*

$a > 1$: a^x increases, **growth**. $0 < a < 1$: it decreases, **decay** (e^{-x} is one). For $k\,a^x + c$: asymptote $y = c$, y-intercept $k + c$, range $y > c$ when $k > 0$.

Example 2

Does $f(x) = 4 \times 0.7^x$ show growth or decay? Write down its y-intercept and the percentage change at each step.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Compare the base with 1	$0.7 < 1$: decay
Step 2 y-intercept: $x = 0$	$f(0) = 4$
Step 3 Percentage: $(1 - 0.7) \times 100\,\%$	it decreases by 30 % at each step

APPLY

4 2×1.5^x: growth or decay? y-intercept?

5 $6\,e^{-x}$: growth or decay? y-intercept?

I.3 An exponential moved

Watch out 1

The asymptote of $3 \times 2^x - 5$ is $y = -5$, not $y = 0$, and its range is $y > -5$: a translation moves the asymptote with the graph.

Example 3

Let $f(x) = 2 \times 3^x - 6$. Write down the asymptote, the y-intercept and the range, then find the zero of f.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Asymptote: the constant	$y = -6$
Step 2 y-intercept: $f(0)$	$2 - 6 = -4$
Step 3 Range: above the asymptote ($k > 0$)	$f(x) > -6$
Step 4 Zero: solve $f(x) = 0$	$3^x = 3,\ x = 1$

APPLY

6 $f(x) = 5 \times 2^x - 10$: asymptote, y-intercept, zero.

7 $f(x) = 7 - e^x$: asymptote, range, y-intercept.

I.4 Solving $a^x = b$

Example 4

Solve $5 \times 2^x = 40$, then solve $3^x = 20$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Isolate the power	$2^x = 8$
Step 2 Write b as a power of a when you can	$2^x = 2^3$: $x = 3$
Step 3 Otherwise take logarithms	$x = \log_3 20 = \dfrac{\ln 20}{\ln 3} = 2.73$

APPLY

8 Solve $2^{x+1} = 32$.

9 Solve $e^x = 5$.

10 Solve $4\,e^{2x} = 12$.

I.5 Exponential models

Example 5

A population is modelled by $P(t) = 500\,e^{0.04t}$, t in years. Find the initial population, $P(10)$, and when the population reaches 1000.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Initial value: $t = 0$	$P(0) = 500$
Step 2 Substitute $t = 10$	$500\,e^{0.4} = 745.9\ldots \approx 746$
Step 3 Write the equation, isolate the power	$e^{0.04t} = 2$
Step 4 Take ln	$t = \dfrac{\ln 2}{0.04} = 17.3$ years

APPLY

11 $V(t) = 3000 \times 0.9^t$: find $V(5)$.

12 Same V: when has the value halved?

13 $N(t) = 200 \times 2^{t/3}$: find $N(6)$.

II LOGARITHMIC FUNCTIONS

SL 2.9 · $\log_a x$, $\ln x$, the inverse of the exponential

II.1 $\log_a x$: the power that gives x

Definition 2 *logarithmic function*

For $a > 0$, $a \neq 1$: $y = \log_a x \iff a^y = x$. $\log_a x$ is the inverse of a^x; $\ln x = \log_e x$ is the inverse of e^x.

Example 6

Find $\log_2 32$, $\log_{10} 0.001$ and $\ln e^3$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Ask: the base to what power?	$2^? = 32,\ 10^? = 0.001,\ e^? = e^3$
Step 2 Read the power	$5,\ -3,\ 3$

APPLY

1 $\log_3 81$

2 $\log_5 1$

3 $\log_4 2$

4 $\ln \frac{1}{e}$

II.2 The graph of a logarithm

Property 2 *the graph of the logarithm*

$y = \log_a x$: domain $x > 0$, range $\mathbb{R}$, through $(1, 0)$, vertical asymptote $x = 0$; the reflection of $y = a^x$ in $y = x$. For $y = \ln(x - h)$ the asymptote is $x = h$.

Watch out 2

The argument must be positive: $\ln(x + 2)$ needs $x > -2$. A negative logarithm is fine ($\ln 0.5 < 0$); a negative argument is not.

Example 7

Let $f(x) = \log_2(x + 1)$. Write down the domain and the asymptote, then find the zero of f and $f(7)$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Argument > 0	$x + 1 > 0$: domain $x > -1$
Step 2 Asymptote where the argument is 0	$x = -1$
Step 3 Zero: argument $= 1$	$x + 1 = 1,\ x = 0$
Step 4 Substitute	$f(7) = \log_2 8 = 3$

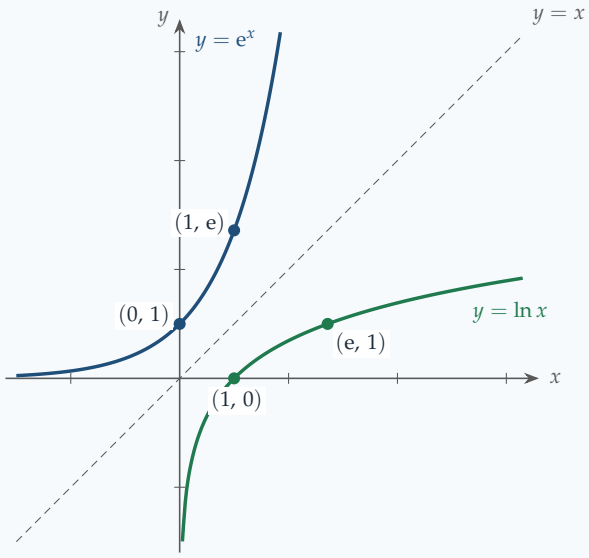


Figure 2 — $y = e^x$ and $y = \ln x$, mirror images in $y = x$: $(0, 1)$ and $(1, 0)$; the asymptote $y = 0$ becomes $x = 0$.

APPLY

5 $\ln(x - 4)$: domain, asymptote, zero.

6 $\log_3(x + 9)$: zero and y-intercept.

II.3 Inverses of exponentials and logarithms

Example 8

Let $f(x) = e^x + 4$. Find $f^{-1}(x)$ and its domain.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Exchange x and y	$x = e^y + 4$
Step 2 Isolate the exponential	$e^y = x - 4$
Step 3 Take ln	$f^{-1}(x) = \ln(x - 4)$
Step 4 Domain = range of f	$x > 4$

APPLY

7 $f(x) = \ln(x + 1)$: find $f^{-1}(x)$.

8 $f(x) = 2^x - 3$: find $f^{-1}(x)$.

9 $f(x) = \log_5 x + 2$: find $f^{-1}(x)$.

II.4 Solving $\log_a(\ldots) = k$

Example 9

Solve $\ln(2x - 1) = 3$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Undo ln with e	$2x - 1 = e^3$
Step 2 Solve for x	$x = \dfrac{e^3 + 1}{2}$
Step 3 Check the argument is positive	$2x - 1 = e^3 > 0$

APPLY

10 Solve $\log_2 x = 5$.

11 Solve $\ln(x + 3) = 0$.

12 Solve $\log_{10} x = -2$.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A Exponential graphs

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 5]

Let $f(x) = 3 \times 2^x - 4$.

(a) Write down the y-intercept of the graph of f. [1]

(b) Write down the equation of the horizontal asymptote of the graph of f. [1]

(c) Write down the range of f. [1]

(d) Solve $f(x) = 20$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 y-intercept: $f(0)$	-1 $3 \times 1 - 4$.	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Asymptote: the constant	$y = -4$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Range: above the asymptote	$f(x) > -4$	A1

WORKING (d)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Isolate the power	$3 \times 2^x = 24 \Rightarrow 2^x = 8$	(M1)
Step 2 Write 8 as a power of 2	$x = 3$	A1

EXAM-STYLE QUESTION 2 Paper 1 · no calculator · [Maximum mark: 6]

The number of bacteria in a dish t hours after the start of an experiment is $N(t) = 200\,e^{kt}$. After 3 hours there are 800 bacteria.

(a) Write down the number of bacteria at the start. [1]

(b) Show that $k = \dfrac{\ln 4}{3}$. [2]

(c) Find the time at which there are 3200 bacteria. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Initial value: $t = 0$	200	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Write the equation, isolate the power	$200\,e^{3k} = 800 \Rightarrow e^{3k} = 4$	M1
Step 2 Take ln	$3k = \ln 4$	A1
Step 3 Divide by 3	$k = \dfrac{\ln 4}{3}$	AG

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Isolate the power	$e^{kt} = 16$	(M1)
Step 2 Take ln	$\dfrac{\ln 4}{3}\,t = \ln 16 = 2\ln 4$	(A1)
Step 3 Solve for t	$t = 6$ hours	A1

$16 = 4^2$: *the population is multiplied by 4 every 3 hours.*

EXAM-STYLE QUESTION 3 Paper 1 · no calculator · [Maximum mark: 6]

Let $f(x) = \ln(x-2) + 1$, for $x > 2$.

(a) Write down the equation of the vertical asymptote of the graph of f. [1]

(b) Find the exact x-intercept of the graph of f. [2]

(c) Find $f^{-1}(x)$. [2]

(d) Write down the range of f^{-1}. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Asymptote: argument $= 0$	$x = 2$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Undo ln with e	$\ln(x-2) = -1 \Rightarrow x - 2 = \mathrm{e}^{-1}$	(M1)
Step 2 Solve for x	$x = 2 + \mathrm{e}^{-1}$	A1
	Exact: not 2.37.	

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Exchange x and y, undo ln	$x = \ln(y-2) + 1 \Rightarrow y - 2 = \mathrm{e}^{x-1}$	(M1)
Step 2 Make y the subject	$f^{-1}(x) = \mathrm{e}^{x-1} + 2$	A1

WORKING (d)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Range of f^{-1} = domain of f	$f^{-1}(x) > 2$	A1
	The domain of f.	

EXAM-STYLE QUESTION 4 Paper 1 · no calculator · [Maximum mark: 4]

Let $f(x) = \log_3(x+1)$, for $x > -1$.

(a) Find $f(8)$. [1]

(b) Find $f^{-1}(x)$. [2]

(c) Write down the range of f^{-1}. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The power of 3 that gives 9	$\log_3 9 = 2$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Exchange x and y, undo $\log_3$	$x = \log_3(y+1) \Rightarrow 3^x = y+1$	(M1)
Step 2 Make y the subject	$f^{-1}(x) = 3^x - 1$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Range of f^{-1} = domain of f	$f^{-1}(x) > -1$	A1

D A model on the GDC (Paper 2)

EXAM-STYLE QUESTION 5 Paper 2 · calculator · [Maximum mark: 6]

The temperature, in °C, of a cup of tea t minutes after it is made is $T(t) = 20 + 70\,e^{-0.08t}$, for $t \geq 0$.

(a) Write down the initial temperature of the tea. [1]

(b) Write down the equation of the horizontal asymptote of the graph of T, and interpret it in context. [1]

(c) Find the time at which the temperature is 40 °C. [2]

(d) Find the time from which the tea stays below 25 °C. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Initial value: $t = 0$	$90\,°C$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Read the asymptote, say what it means	$T = 20$: the tea cools towards the temperature of the room, 20 °C	R1

The equation and its meaning.

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Write the equation	$20 + 70\,e^{-0.08t} = 40$	(M1)
Step 2 Solve on the GDC	$t = 15.7$ min $(15.6595\ldots)$	A1

WORKING (d)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Write the equation	$20 + 70\,e^{-0.08t} = 25$	(M1)
Step 2 Solve on the GDC	after 33.0 min $(32.9882\ldots)$	A1

IV WHICH METHOD? READ THE QUESTION

2.10 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"the asymptote" of $k\,a^x + c$	the constant added	$y = c$, as an equation P1
"the range" of an exponential	the asymptote and the sign of k	$y > c$ (or $y < c$ when $k < 0$) P1
"the domain" of a logarithm	the argument is positive	solve "inside > 0" W2
"solve" $a^x = b$ *or* $e^{kx} = b$	take logarithms	$x = \log_a b$, or $kx = \ln b$ D2
"find k*"* in $A\,e^{kt}$	one value of the model	divide by A, then ln P1
"find f^{-1}*"* of e *or* ln	they undo each other	ln becomes $e^{\cdots}$ and back; asymptotes swap P2
Paper 2, "how long until…"	an equation in t	the GDC: the model and the target value, then the intersection

In the exam 1 *command terms*

Exact: keep e and ln, no decimals. **Asymptote**: an equation. **Initial value**: $t = 0$. **Show that**: start from the given data, end at the value given, every step written.

V ON THE GDC: PAPER 2

SL 2.9 · 2.10 · NumWorks · every calculator move of this topic

1 Logarithms in any base

Calculation → Toolbox → log(x, a)

$\log_a b$ directly, or as $\dfrac{\ln b}{\ln a}$. **On paper:** $3^x = 20$, so $x = \log_3 20 = 2.73$.

CALCULATION

```
log(20,3)
2.726833028
ln(16)/ln(4)*3
6
```

2 When does the model reach a value?

Grapher → T(x), y = 40 → Find → Intersection

Draw the model and the target value; the intersection gives the time. **On paper:** "$20 + 70e^{-0.08t} = 40$", then $t = 15.7$ min.

GRAPHER · INTERSECTION

```
f(x)=20+70e^(-0.08x) g(x)=40
x=15.65954 y=40
```

ANSWERS TO THE APPLY QUESTIONS

I.1 $\frac{1}{25}$

I.2 8

I.3 $y > 0$

I.4 growth; 2

I.5 decay; 6

I.6 $y = -10$; -5; $x = 1$

I.7 $y = 7$; $f(x) < 7$; 6

I.8 $x = 4$

I.9 $x = \ln 5$

I.10 $x = \frac{1}{2}\ln 3$

I.11 1770 (1771.47)

I.12 $t = 6.58$

I.13 800

II.1 4

II.2 0

II.3 $\frac{1}{2}$

II.4 -1

II.5 $x > 4$; $x = 4$; $x = 5$

II.6 $x = -8$; 2

II.7 $e^x - 1$

II.8 $\log_2(x+3)$

II.9 5^{x-2}

II.10 $x = 32$

II.11 $x = -2$

II.12 $x = 0.01$

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