# Quadratic functions



One parabola, three ways of writing it: each form answers one question. Then the discriminant says, before any solving, how many times the parabola crosses the $x$-axis.

## I THE THREE FORMS OF A QUADRATIC

SL 2.6 · $ax^2+bx+c$, $a(x-p)(x-q)$, $a(x-h)^2+k$

### I.1 The expanded form

**Definition 1** *quadratic function*

$f(x)=ax^2+bx+c$, with $a\neq 0$. Its graph is a **parabola**: it opens upwards if $a>0$, downwards if $a<0$, and is symmetric about its **axis**, the vertical line through the **vertex**.

**Property 1** *what the expanded form tells*

$y=ax^2+bx+c$: the $y$-intercept is $c$; the axis of symmetry is $x=-\dfrac{b}{2a}$.

**Example 1**

Find the axis of symmetry and the vertex of the graph of $y=2x^2+12x+7$.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Read $a$, $b$, $c$ | $a=2,\ b=12,\ c=7$ |
| **Step 2** Axis: $x=-\dfrac{b}{2a}$<br>Keep the minus sign. | $x=-\dfrac{12}{4}=-3$ |
| **Step 3** Substitute to find $y$ | $y=2(-3)^2+12(-3)+7=-11$ |
| **Step 4** Write the vertex as a point | vertex $(-3,\ -11)$ |

**APPLY**

1 $y=x^2-10x+4$: axis and vertex.

2 $y=-3x^2+6x+1$: axis and vertex.

3 $y=5-4x-x^2$: axis, vertex and $y$-intercept.

## I.2 The factorised form

**Property 2** *what the factorised form tells*

$y = a(x-p)(x-q)$: the zeros are $p$ and $q$; the axis is halfway between them, $x = \dfrac{p+q}{2}$.

**Example 2**

Find the zeros, the axis of symmetry and the vertex of the graph of $y = 3(x+2)(x-4)$.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Read the zeros from the factors<br>$(x+2)$ gives $x=-2$: the sign changes. | $x = -2$ and $x = 4$ |
| **Step 2** Axis halfway between the zeros | $x = \dfrac{-2+4}{2} = 1$ |
| **Step 3** Substitute to find $y$ | $y = 3(3)(-3) = -27$, vertex $(1, -27)$ |

**APPLY**

**4** $y = -(x-1)(x-7)$: zeros, axis, vertex.

**5** $y = 2(x+5)(x+1)$: zeros, axis, vertex.

**6** $y = x(x-6)$: zeros, axis, vertex.

## I.3 The vertex form: completing the square

**Property 3** *what the vertex form tells*

$y = a(x-h)^2 + k$: the vertex is $(h, k)$, the axis is $x = h$. To get it when $a = 1$:

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c.$$

**Example 3**

Write $x^2 - 6x + 5$ in the form $(x-h)^2 + k$, and give the vertex of its graph.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Halve $b$ | $\dfrac{-6}{2} = -3$ |
| **Step 2** Square the bracket, subtract the square | $(x-3)^2 - 9 + 5$ |
| **Step 3** Tidy the constant | $(x-3)^2 - 4$ |
| **Step 4** Read the vertex $(h, k)$<br>$(x-3)$ gives $h = +3$. | vertex $(3, -4)$ |

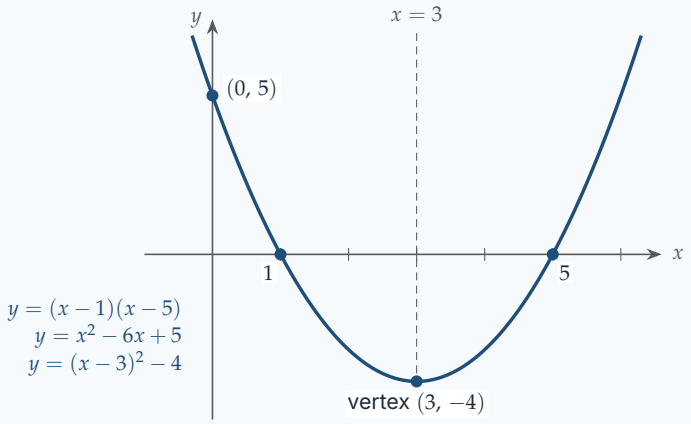



*Figure 1 — Example 3 on the graph: $y = (x-1)(x-5) = x^2 - 6x + 5 = (x-3)^2 - 4$ — zeros 1 and 5, y-intercept 5, vertex $(3, -4)$.*

**APPLY**

**7** Write $x^2 - 4x + 9$ in vertex form; give the vertex.

**8** Write $x^2 + 10x + 18$ in vertex form; give the vertex.

**9** Write $x^2 + 3x + 1$ in vertex form.

## I.4 Completing the square when $a \neq 1$

**Watch out 1**

Take $a$ out of the $x$ terms only. The square you subtract is inside the bracket: it is multiplied by $a$ too.

**Example 4**

Write $3x^2 + 18x + 20$ in the form $a(x-h)^2 + k$.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Take $a$ out of the $x$ terms | $3\left(x^2 + 6x\right) + 20$ |
| **Step 2** Complete the square inside | $3\left[(x+3)^2 - 9\right] + 20$ |
| **Step 3** Multiply out by $a$ | $3(x+3)^2 - 27 + 20$ |
| The $-9$ becomes $-27$: this is where marks are lost. | |
| **Step 4** Tidy the constant | $3(x+3)^2 - 7$, vertex $(-3, -7)$ |

**APPLY**

**10** Write $2x^2 + 8x + 3$ in the form $a(x-h)^2 + k$.

**11** Write $-x^2 + 6x - 4$ in the form $a(x-h)^2 + k$.

**12** Write $4x^2 - 8x + 1$ in the form $a(x-h)^2 + k$.

## I.5 Finding the equation from the graph

**Example 5**

A parabola crosses the $x$-axis at $x = 2$ and $x = 6$, and the $y$-axis at $(0, -36)$. Find its equation.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Write the factors from the zeros | $y = a(x-2)(x-6)$ |
| **Step 2** Substitute the third point | $-36 = a(-2)(-6) = 12a$ |
| **Step 3** Solve for $a$ | $a = -3$ |
| **Step 4** Write the equation | $y = -3(x-2)(x-6)$ |

**APPLY**

**13** Zeros $-3$ and $1$, through $(0, -12)$.

**14** Vertex $(2, 5)$, through $(4, -7)$. Start from $a(x-h)^2 + k$.

**15** Zeros $0$ and $4$, through $(2, 8)$.

# II EQUATIONS AND THE DISCRIMINANT

SL 2.7 · the quadratic formula, $\Delta = b^2 - 4ac$ and the number of roots

## II.1 The quadratic formula

**Property 4** *the quadratic formula*

The solutions of $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Factorise first when you can; the formula always works.

**Example 6**

Solve $2x^2 - 3x - 4 = 0$. Give your answers correct to three significant figures.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Read $a, b, c$ | $a = 2,\ b = -3,\ c = -4$ |
| **Step 2** Compute $b^2 - 4ac$ first<br>$-4ac = -4(2)(-4) = +32$. | $9 + 32 = 41$ |
| **Step 3** Use the formula | $x = \dfrac{3 \pm \sqrt{41}}{4}$ |
| **Step 4** Round at the very end | $x = 2.35$ or $x = -0.851$ |

**APPLY**

**1** Solve $x^2 - 7x + 10 = 0$.

**2** Solve $x^2 + 2x - 5 = 0$, exactly.

**3** Solve $3x^2 + 5x - 1 = 0$ (3 s.f.).

## II.2 The discriminant: how many roots

**Definition 2** *discriminant*

$\Delta = b^2 - 4ac$. $\Delta > 0$: two distinct real roots; $\Delta = 0$: two equal roots, the parabola touches the axis; $\Delta < 0$: no real roots.

**Example 7**

How many real roots does $x^2 - 4x + 7 = 0$ have?

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Read $a, b, c$ | $a = 1,\ b = -4,\ c = 7$ |
| **Step 2** Compute $\Delta$ | $\Delta = 16 - 28 = -12$ |
| **Step 3** Compare $\Delta$ with 0 | $\Delta < 0$ |
| **Step 4** Conclude in words | no real roots |

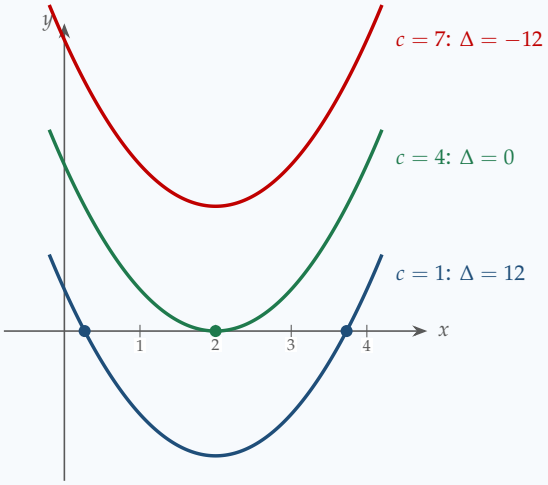



*Figure 2 — Example 7 is $c = 7$: $\Delta = -12$, no root. $y = x^2 - 4x + c$ for $c = 1$, $c = 4$ and $c = 7$: $\Delta = 12$, $0$ and $-12$.*

APPLY

**4** $x^2 - 6x + 9 = 0$: how many real roots?

**5** $2x^2 + x - 6 = 0$: how many real roots?

**6** $5x^2 - 2x + 1 = 0$: how many real roots?

## II.3 A parameter $k$ in the equation

**Watch out 2**

Read the words: "two distinct real roots" is $\Delta > 0$; "two equal roots" or "touches" is $\Delta = 0$; "real roots" is $\Delta \geq 0$; "no real roots" or "never meets" is $\Delta < 0$.

**Example 8**

The equation $x^2 + kx + 16 = 0$ has two equal roots. Find the possible values of $k$.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Write $\Delta$ with $k$ in it | $\Delta = k^2 - 4(1)(16) = k^2 - 64$ |
| **Step 2** Translate the words | equal roots: $k^2 - 64 = 0$ |
| **Step 3** Solve: both signs | $k = 8$ or $k = -8$ |
| $k^2 = 64$ has two solutions. | |

APPLY

**7** $2x^2 - 4x + k = 0$ has two distinct real roots. Find $k$.

**8** $x^2 - 2x + k - 3 = 0$ has no real roots. Find $k$.

**9** $x^2 + (k+1)x + 4 = 0$ has two equal roots. Find $k$.

# III QUADRATIC INEQUALITIES

SL 2.7 · solve $ax^2 + bx + c > 0$ with a sketch

## III.1 Solving with a sketch

**Property 5** *solving with a sketch*

Find the roots, sketch the parabola, read where it is above ($> 0$) or below ($< 0$) the $x$-axis. For $a > 0$ and roots $p < q$: below between the roots, above outside them.

**Watch out 3**

$x^2 > 9$ is not $x > 3$ alone: it is $x < -3$ or $x > 3$.

**Example 9**

Solve $x^2 + x - 12 < 0$.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Solve the equation $= 0$ | $(x+4)(x-3) = 0$: $x = -4, x = 3$ |
| **Step 2** Sketch: $a > 0$, opens upwards | the parabola is below the axis between the roots (Figure 3) |
| **Step 3** Read the answer | $-4 < x < 3$ |

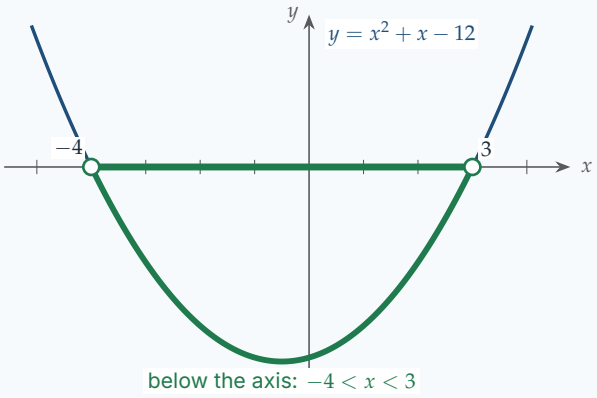



*Figure 3 — Example 9: $y = x^2 + x - 12$ is below the x-axis exactly for $-4 < x < 3$.*

**APPLY**

1 Solve $x^2 - 9x + 14 \geq 0$.

2 Solve $-x^2 + 4x + 12 > 0$.

3 Solve $x^2 > 16$.

## III.2 An inequality for $k$

**Example 10**

The equation $x^2 - kx + 25 = 0$ has no real roots. Find the possible values of $k$.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Write $\Delta$ with $k$ in it | $\Delta = k^2 - 100$ |
| **Step 2** Translate: no real roots | $k^2 - 100 < 0$ |
| **Step 3** Solve with a sketch of $k^2 - 100$<br>Between the roots: the parabola in $k$ opens upwards. | $-10 < k < 10$ |

**APPLY**

**4** $3x^2 + kx + 3 = 0$ has two distinct real roots. Find $k$.

**5** $x^2 + 2kx + 9 = 0$ has no real roots. Find $k$.

# IV EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

## A Forms and the vertex

**EXAM-STYLE QUESTION 1** Paper 1 · no calculator · [Maximum mark: 6]

Let $f(x) = 2x^2 - 8x + 5$.

(a) Write $f(x)$ in the form $a(x-h)^2 + k$. [3]

(b) Write down the coordinates of the vertex of the graph of $f$. [1]

(c) Write down the equation of the axis of symmetry and the $y$-intercept of the graph of $f$. [2]

**WORKING (a)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Take $a$ out of the $x$ terms | $2\left(x^2 - 4x\right) + 5$ | (M1) |
| **Step 2** Complete the square, multiply out by $a$ | $2(x-2)^2 - 8 + 5$ | (A1) |
| | *The $-4$ inside the bracket becomes $-8$.* | |
| **Step 3** Tidy the constant | $f(x) = 2(x-2)^2 - 3$ | A1 |

**WORKING (b)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Read the vertex $(h, k)$ | $(2, -3)$ | A1 |

**WORKING (c)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Axis: $x = h$ | $x = 2$ | A1 |
| | *An equation: "2" alone is not the axis.* | |
| **Step 2** $y$-intercept: $f(0)$ | $5$ | A1 |

**EXAM-STYLE QUESTION 2** Paper 1 · no calculator · [Maximum mark: 6]

The graph of a quadratic function $f$ crosses the $x$-axis at $(-1,\ 0)$ and $(3,\ 0)$, and the $y$-axis at $(0,\ 6)$.

(a) Find $f(x)$, in the form $a(x-p)(x-q)$. [3]

(b) Find the coordinates of the vertex of the graph of $f$. [2]

(c) Write down the range of $f$. [1]

**WORKING (a)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Write the factors from the zeros | $f(x) = a(x+1)(x-3)$ | (M1) |
| **Step 2** Substitute the third point $(0,\ 6)$ | $a(1)(-3) = 6$ | M1 |
| **Step 3** Solve for $a$ | $f(x) = -2(x+1)(x-3)$ | A1 |

**WORKING (b)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Axis halfway between the zeros | $x = \dfrac{-1+3}{2} = 1$ | (A1) |
| **Step 2** Substitute to find $y$ | $(1,\ 8)$ | A1 |

**WORKING (c)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Read the range from the vertex | $f(x) \leq 8$ | A1 |

*$a < 0$: the vertex is the highest point.*

## B The discriminant

**EXAM-STYLE QUESTION 3** — Paper 1 · no calculator · [Maximum mark: 4]

The equation $x^2 + (k-2)x + 4 = 0$ has two equal real roots.

(a) Find the possible values of $k$. [4]

**WORKING (a)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Write $\Delta$ with $k$ in it | $\Delta = (k-2)^2 - 4 \times 1 \times 4$ | (M1) |
| **Step 2** Equal roots: $\Delta = 0$ | $(k-2)^2 - 16 = 0$ | M1 |
| **Step 3** Take the square root | $k - 2 = \pm 4$<br>*Both signs.* | (A1) |
| **Step 4** Solve for $k$ | $k = -2$ or $k = 6$ | A1 |

**EXAM-STYLE QUESTION 4** — Paper 1 · no calculator · [Maximum mark: 6]

The line $y = mx - 1$ meets the curve $y = x^2 + 3$ at exactly one point.

(a) Show that $x^2 - mx + 4 = 0$. [1]

(b) Find the possible values of $m$. [3]

(c) For the positive value of $m$, find the coordinates of the point where the line meets the curve. [2]

**WORKING (a)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Equate the two expressions for $y$ | $x^2 + 3 = mx - 1$ | M1 |
| **Step 2** Rearrange to the given form | $x^2 - mx + 4 = 0$ | AG |

**WORKING (b)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Write $\Delta$ with $m$ in it | $\Delta = m^2 - 16$ | (A1) |
| **Step 2** One point of contact: $\Delta = 0$ | $m^2 - 16 = 0$ | M1 |
| **Step 3** Solve for $m$ | $m = \pm 4$ | A1 |

**WORKING (c)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Put $m = 4$ in the equation and solve | $x^2 - 4x + 4 = 0 \Rightarrow x = 2$ | (A1) |
| **Step 2** Find $y$ | $(2, 7)$<br>*$y$ from the line: $4(2) - 1 = 7$.* | A1 |

## C The discriminant and an inequality

**EXAM-STYLE QUESTION 5** — Paper 1 · no calculator · [Maximum mark: 5]

The equation $kx^2 - 4x + k = 0$, where $k \neq 0$, has no real roots.

(a) Find the possible values of $k$. [5]

**WORKING (a)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Write $\Delta$ with $k$ in it | $\Delta = 16 - 4k^2$ | (A1) |
| **Step 2** No real roots: $\Delta < 0$ | $16 - 4k^2 < 0$ | M1 |
| **Step 3** Isolate $k^2$ | $k^2 > 4$ | (A1) |
| **Step 4** Solve with a sketch: outside the roots | $k < -2$ or $k > 2$ | A1A1 |

*A1 for each inequality; $-2 < k < 2$ earns A0A0.*

## D In context, with the GDC (Paper 2)

**EXAM-STYLE QUESTION 6** — Paper 2 · calculator · [Maximum mark: 6]

A ball is thrown upwards. Its height, in metres, $t$ seconds after it is thrown is $h(t) = -4.9t^2 + 12t + 1.5$, for $t \geq 0$, until it hits the ground.

(a) Find the maximum height of the ball. [2]

(b) Find the time at which the ball hits the ground. [2]

(c) Find the length of time for which the ball is more than 5 metres above the ground. [2]

**WORKING (a)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Find when the vertex is reached: $-\frac{b}{2a}$, or the maximum on the GDC | $t = \frac{12}{9.8} = 1.22\ldots$ | (M1) |
| **Step 2** Height at the vertex | 8.85 m $(8.84693\ldots)$ | A1 |

**WORKING (b)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Set $h(t) = 0$ | $-4.9t^2 + 12t + 1.5 = 0$ | (M1) |
| **Step 2** Solve on the GDC, keep $t \geq 0$ | $t = 2.57$ s $(2.56817\ldots)$ | A1 |

*Reject the negative root.*

**WORKING (c)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Set $h(t) = 5$ and solve | $-4.9t^2 + 12t + 1.5 = 5$: $t = 0.338$ or $t = 2.11$ | (M1) |
| **Step 2** Subtract the two times | $2.11 - 0.338 = 1.77$ s $(1.77210\ldots)$ | A1 |

## V WHICH METHOD? READ THE QUESTION

2.7 · the whole topic on one page

| THE QUESTION SAYS | YOU KNOW | YOU WRITE |
|---|---|---|
| *"write in the form* $a(x-h)^2+k$*"* | complete the square | take $a$ out, halve $b$, square and subtract P3 |
| *"the vertex", "the axis of symmetry"* | $x=-\frac{b}{2a}$, or $(h, k)$ | the vertex as a point; the axis as $x=\ldots$ P1 |
| *three points, or zeros and a point* | $a(x-p)(x-q)$ | the zeros give the factors; one more point gives $a$ P2 |
| *"solve"* $ax^2+bx+c=0$ | factorise or the formula | $x=\frac{-b\pm\sqrt{\Delta}}{2a}$ P4 |
| *"two equal roots", "touches", "tangent"* | $\Delta=0$ | an equation in the parameter D2 |
| *"no real roots", "never meets"* | $\Delta<0$ | an inequality in the parameter; a sketch to solve it W2 |
| *"solve* $ax^2+bx+c>0$*"* | the roots and a sketch | between the roots, or outside them P5 |

**In the exam 1** *command terms*

**Two distinct real roots**: $\Delta>0$. **Equal roots**: $\Delta=0$. **Real roots**: $\Delta\geq 0$. **Show that**: every step, ending with the result given. **Write down the axis**: an equation, $x=\ldots$.

## VI ON THE GDC: PAPER 2

SL 2.6 · 2.7 · NumWorks · every calculator move of this topic

### 1 Solve a quadratic equation

Equations → Add an equation → −4.9x²+12x+1.5=0 → Solve

The Equations app solves a polynomial equation and gives Δ. **On paper:** the equation, then the root that makes sense: $t=2.57$.

EQUATIONS · POLYNOMIAL

```
-4.9x^2+12x+1.5=0
Delta=173.4
x1=-0.1191983
x2=2.568178
```

## 2 The vertex: maximum or minimum

Grapher → f(x) → Find → Maximum

The vertex of a parabola is its maximum ($a < 0$) or minimum ($a > 0$). **On paper:** "maximum height 8.85 m, at $t = 1.22$ s".

GRAPHER · MAXIMUM

```
f(x)=-4.9x^2+12x+1.5
x=1.22449 y=8.846939
```

### ANSWERS TO THE APPLY QUESTIONS

**I.1** $x = 5$; $(5, -21)$

**I.2** $x = 1$; $(1, 4)$

**I.3** $x = -2$; $(-2, 9)$; $(0, 5)$

**I.4** 1 and 7; $x = 4$; $(4, 9)$

**I.5** $-5$ and $-1$; $x = -3$; $(-3, -8)$

**I.6** 0 and 6; $x = 3$; $(3, -9)$

**I.7** $(x-2)^2 + 5$; $(2, 5)$

**I.8** $(x+5)^2 - 7$; $(-5, -7)$

**I.9** $\left(x + \frac{3}{2}\right)^2 - \frac{5}{4}$

**I.10** $2(x+2)^2 - 5$

**I.11** $-(x-3)^2 + 5$

**I.12** $4(x-1)^2 - 3$

**I.13** $y = 4(x+3)(x-1)$

**I.14** $y = -3(x-2)^2 + 5$

**I.15** $y = -2x(x-4)$

**II.1** $x = 2$ or $x = 5$

**II.2** $x = -1 \pm \sqrt{6}$

**II.3** $x = 0.180$ or $x = -1.85$

**II.4** $\Delta = 0$: two equal roots $(x = 3)$

**II.5** $\Delta = 49 > 0$: two distinct real roots

**II.6** $\Delta = -16 < 0$: none

**II.7** $16 - 8k > 0$, so $k < 2$

**II.8** $4 - 4(k-3) < 0$, so $k > 4$

**II.9** $k = 3$ or $k = -5$

**III.1** $x \le 2$ or $x \ge 7$

**III.2** $-2 < x < 6$

**III.3** $x < -4$ or $x > 4$

**III.4** $k^2 - 36 > 0$: $k < -6$ or $k > 6$

**III.5** $4k^2 - 36 < 0$: $-3 < k < 3$

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