Rational functions

Divide by x and the graph breaks in two: near $x = 0$ the values race off to infinity, far away they fade to nothing. Every rational function of the syllabus is this hyperbola, moved and stretched; two lines, the asymptotes, hold its whole shape.

I THE RECIPROCAL FUNCTION

SL 2.8 · $f(x) = \frac{1}{x}$, its graph and its asymptotes

I.1 The graph of $\frac{1}{x}$

Definition 1 *reciprocal function*

$f(x) = \dfrac{1}{x}$, for $x \neq 0$. Its graph is a **hyperbola** in two branches, with the vertical **asymptote** $x = 0$ and the horizontal asymptote $y = 0$: the graph comes as close as we like to these lines without meeting them.

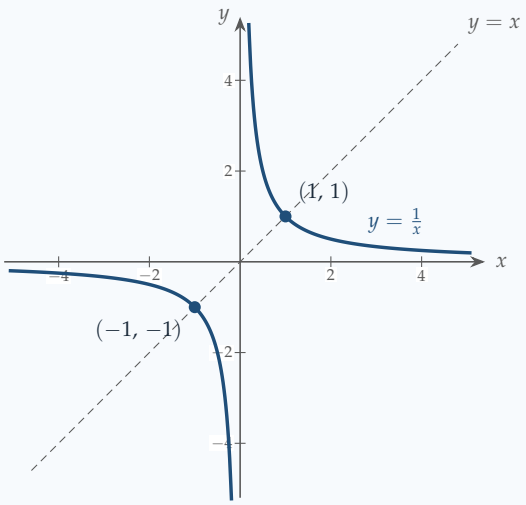


Figure 1 — $y = \dfrac{1}{x}$ and its two asymptotes: Example 1's points $\left(4, \frac{1}{4}\right)$ and $\left(\frac{1}{8}, 8\right)$ hug the x-axis and the y-axis.

Watch out 1

$\dfrac{1}{x}$ is not defined at $x = 0$, and 0 is never one of its values: domain $x \neq 0$, range $y \neq 0$.

Example 1

Let $f(x) = \dfrac{1}{x}$. Find $f(4)$ and $f(-0.5)$, then solve $f(x) = 8$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Substitute	$f(4) = \dfrac{1}{4},\ f(-0.5) = -2$
Step 2 Write the equation	$\dfrac{1}{x} = 8$
Step 3 Multiply by x, divide by 8	$x = \dfrac{1}{8}$

APPLY

1 $f(x) = \dfrac{1}{x}$: find $f(0.01)$.

2 Solve $\dfrac{1}{x} = -5$.

3 Domain and range of $\dfrac{1}{x}$.

I.2 $\frac{k}{x}$ is its own inverse

Property 1 *self-inverse*

$f(f(x)) = \dfrac{1}{1/x} = x$: $\dfrac{1}{x}$ is its own inverse, and its graph is symmetric about $y = x$ and about $y = -x$.

Example 2

Let $f(x) = \dfrac{4}{x}$. Find $f^{-1}(x)$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Exchange x and y	$x = \dfrac{4}{y}$
Step 2 Make y the subject	$y = \dfrac{4}{x}$
Step 3 Conclude	$f^{-1}(x) = \dfrac{4}{x}$: f is its own inverse

APPLY

4 $f(x) = -\dfrac{3}{x}$: find $f^{-1}(x)$.

5 Solve $\dfrac{1}{x} = x$.

II.1 The asymptotes

Property 2 *asymptotes and intercepts*

For $f(x) = \dfrac{ax+b}{cx+d}$, $c \neq 0$: **vertical asymptote** $x = -\dfrac{d}{c}$ (denominator 0); **horizontal asymptote** $y = \dfrac{a}{c}$ (ratio of the x-coefficients); x-intercept $-\dfrac{b}{a}$; y-intercept $\dfrac{b}{d}$.

Example 3

Write down the equations of the asymptotes of the graph of $f(x) = \dfrac{4x-1}{2x+6}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Vertical: denominator $= 0$	$2x+6=0$: $x=-3$
Step 2 Horizontal: ratio of the x-coefficients	$y = \dfrac{4}{2} = 2$
Step 3 Write them as equations	$x=-3$ and $y=2$

APPLY

1 Asymptotes of $\dfrac{x+5}{x-4}$.

2 Asymptotes of $\dfrac{6x}{3x-1}$.

3 Asymptotes of $\dfrac{2-x}{x+1}$.

II.2 The intercepts

Example 4

Find the intercepts of the graph of $f(x) = \dfrac{3x-6}{x+4}$ with the axes.

WHAT YOU DO	WHAT YOU WRITE
Step 1 x-intercept: numerator $= 0$	$3x-6=0$: $(2,\,0)$
Step 2 y-intercept: $f(0)$	$\dfrac{-6}{4} = -\dfrac{3}{2}$: $\left(0,\,-\dfrac{3}{2}\right)$
Step 3 Write both as points	$(2,\,0)$ and $\left(0,\,-\frac{3}{2}\right)$

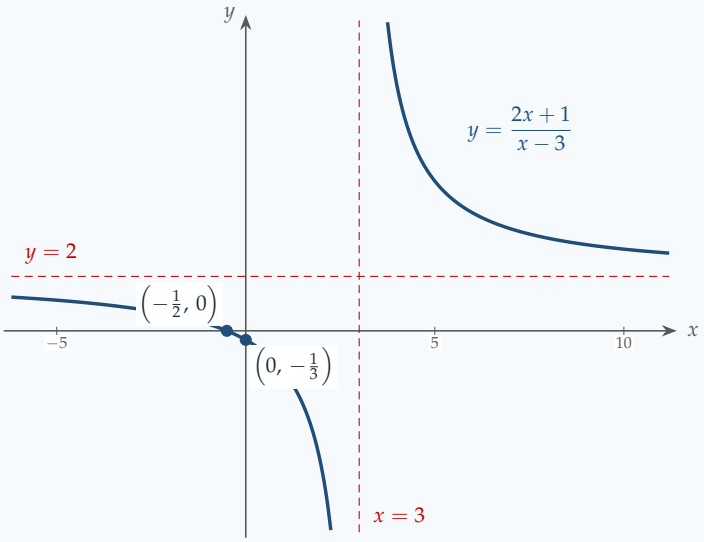


Figure 2 — Exam-style question 1 on the graph: $f(x) = \dfrac{2x+1}{x-3}$, asymptotes $x = 3$ and $y = 2$, intercepts $\left(-\frac{1}{2}, 0\right)$ and $\left(0, -\frac{1}{3}\right)$.

APPLY

4 Intercepts of $\dfrac{2x+8}{x-1}$.

5 Intercepts of $\dfrac{5x-10}{2x+5}$.

II.3 Domain and range

Property 3 *domain and range*

Domain of $\dfrac{ax+b}{cx+d}$: $x \neq -\dfrac{d}{c}$; range: $y \neq \dfrac{a}{c}$.

Watch out 2

The horizontal asymptote is $\dfrac{a}{c}$, not $\dfrac{b}{d}$ (that is the y-intercept). On a sketch, the asymptotes are dashed and labelled with their equations.

Example 5

Write down the domain and the range of $f(x) = \dfrac{x-7}{2x+3}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Domain: leave out the vertical asymptote	$x \neq -\dfrac{3}{2}$
Step 2 Range: leave out the horizontal asymptote	$f(x) \neq \dfrac{1}{2}$

APPLY

6 Domain and range of $\dfrac{4x+1}{x-5}$.

7 Domain and range of $\dfrac{3-2x}{x+6}$.

II.4 $\frac{1}{x}$ moved and stretched

Example 6

Show that $4 - \dfrac{2}{x+1} = \dfrac{4x+2}{x+1}$, and write down the asymptotes.

WHAT YOU DO	WHAT YOU WRITE
Step 1 One denominator	$\dfrac{4(x+1)-2}{x+1}$
Step 2 Simplify the numerator	$= \dfrac{4x+2}{x+1}$
Step 3 Read the asymptotes The constant 4 is the horizontal asymptote.	$x = -1$ and $y = 4$

APPLY

8 Write $1 + \dfrac{6}{x-2}$ as one fraction.

9 Write $\dfrac{3x+5}{x+1}$ as $3 + \dfrac{k}{x+1}$.

II.5 Coefficients from the asymptotes

Example 7

The graph of $f(x) = \dfrac{ax+2}{x+b}$ has asymptotes $x = -4$ and $y = 5$. Find a and b.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Vertical: the denominator is 0 at $x = -4$	$-4 + b = 0$: $b = 4$
Step 2 Horizontal: ratio of the x-coefficients	$\dfrac{a}{1} = 5$: $a = 5$
Step 3 Check a point	$f(0) = \dfrac{2}{4} = \dfrac{1}{2}$

APPLY

10 $f(x) = \dfrac{ax-1}{2x+b}$, asymptotes $x = 3$, $y = -2$: find a and b.

11 $f(x) = \dfrac{ax+b}{x-1}$, asymptote $y = 2$, through $(0, 5)$: find a and b.

II.6 The inverse of a rational function

Example 8

Let $f(x) = \dfrac{x+5}{x-3}$, for $x \neq 3$. Find $f^{-1}(x)$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Exchange x and y	$x = \dfrac{y+5}{y-3}$
Step 2 Multiply out, collect the y terms	$xy - 3x = y + 5,\ y(x-1) = 3x+5$
Step 3 Divide	$f^{-1}(x) = \dfrac{3x+5}{x-1}$

The asymptotes swap: $x = 1$, $y = 3$.

APPLY

12 $f(x) = \dfrac{2x-3}{x+4}$: find $f^{-1}(x)$.

II.7 A line and a hyperbola on the GDC

Example 9

Let $f(x) = \dfrac{x+3}{x+1}$ and $g(x) = 2x$. Solve $f(x) > g(x)$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Intersections on the GDC	$x = -1.5$ and $x = 1$
Step 2 Mark the vertical asymptote	$x = -1$ splits the graph of f in two
Step 3 Read where f is above g	left of -1.5, and between the asymptote and 1
Step 4 Write the solution	$x < -1.5$ or $-1 < x < 1$

The asymptote is a boundary too.

APPLY

13 Solve $\dfrac{x+3}{x-2} = x$.

14 Solve $\dfrac{2}{x+1} = 3 - x$.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A Asymptotes, intercepts, domain and range

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 6]

Let $f(x) = \dfrac{2x+1}{x-3}$, for $x \neq 3$.

(a) Write down the equation of the vertical asymptote and the equation of the horizontal asymptote of the graph of f. [2]

(b) Find the coordinates of the points where the graph of f meets the axes. [2]

(c) Write down the range of f. [1]

(d) Solve $f(x) = 3$. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Vertical: denominator $= 0$	$x = 3$	A1
Step 2 Horizontal: ratio of the x-coefficients	$y = 2$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 x-intercept: numerator $= 0$	$\left(-\frac{1}{2}, 0\right)$	A1
Step 2 y-intercept: $f(0)$	$\left(0, -\frac{1}{3}\right)$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Range: leave out the horizontal asymptote	$f(x) \neq 2$	A1

The horizontal asymptote is the value never taken.

WORKING (d)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Write the equation and solve	$2x + 1 = 3x - 9 \Rightarrow x = 10$	A1

B Coefficients and the inverse

EXAM-STYLE QUESTION 2 Paper 1 · no calculator · [Maximum mark: 6]

Let $f(x) = \dfrac{ax+3}{x+b}$. The graph of f has a vertical asymptote $x = -2$ and a horizontal asymptote $y = 4$.

(a) Write down the value of a and the value of b. [2]

(b) Find $f^{-1}(x)$. [3]

(c) Write down the equations of the asymptotes of the graph of f^{-1}. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Read the coefficients from the asymptotes	$b = 2$	A1
	The denominator is 0 at $x = -2$.	
	$a = 4$	A1
	$\frac{a}{1} = 4$.	

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Exchange x and y	$x = \dfrac{4y+3}{y+2}$	(M1)
Step 2 Collect the y terms on one side	$xy + 2x = 4y + 3 \Rightarrow y(x-4) = 3 - 2x$	(A1)
Step 3 Divide	$f^{-1}(x) = \dfrac{3-2x}{x-4}$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The asymptotes swap	$x = 4$ and $y = -2$	A1
	The asymptotes swap, like every point: reflection in $y = x$.	

C A hyperbola moved and stretched

EXAM-STYLE QUESTION 3 Paper 1 · no calculator · [Maximum mark: 5]

Let $f(x) = 3 + \dfrac{2}{x-1}$, for $x \neq 1$.

(a) Write down the equations of the asymptotes of the graph of f. [2]

(b) Show that $f(x) = \dfrac{3x-1}{x-1}$. [2]

(c) Find the x-intercept of the graph of f. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Read the asymptotes	$x = 1$	A1
	$y = 3$	A1
	$\frac{2}{x-1} \to 0$ *far from the origin.*	

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 One denominator	$\dfrac{3(x-1)+2}{x-1}$	M1A1
	A common denominator, then simplify.	
Step 2 Simplify to the given form	$= \dfrac{3x-1}{x-1}$	AG

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Numerator $= 0$	$x = \dfrac{1}{3}$	A1
	The numerator is 0.	

D With the GDC (Paper 2)

EXAM-STYLE QUESTION 4 Paper 2 · calculator · [Maximum mark: 6]

Let $f(x) = \dfrac{x+4}{2x-3}$, for $x \neq 1.5$, and $g(x) = x - 1$.

(a) Write down the equations of the asymptotes of the graph of f. [2]

(b) Find the x-coordinates of the points where the graphs of f and g intersect. [2]

(c) Hence solve $f(x) > g(x)$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Read the asymptotes	$x = 1.5$	A1
	$y = 0.5$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Intersections on the GDC	$x = -0.158 \ \ (-0.158312\ldots)$	A1
	$x = 3.16 \ \ (3.15831\ldots)$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Left of the first intersection	$x < -0.158$	A1
	Left of the first intersection, on the left branch.	
Step 2 Between the asymptote and the second	$1.5 < x < 3.16$	A1
	The asymptote $x = 1.5$ starts the second interval.	

IV WHICH METHOD? READ THE QUESTION

2.8 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"write down the asymptotes"	denominator $= 0$; ratio of the x-terms	$x = -\frac{d}{c}$ and $y = \frac{a}{c}$, as equations P2
"the intercepts"	$x = 0$; numerator $= 0$	$\left(0, \frac{b}{d}\right)$ and $\left(-\frac{b}{a}, 0\right)$, as points P2
"the domain", "the range"	the asymptotes	$x \neq -\frac{d}{c}$ and $y \neq \frac{a}{c}$ P3
asymptotes given, coefficients unknown	c, d from the vertical; a from the horizontal	read them, then check a point P2
"find f^{-1}" of a rational function	exchange x and y	collect the y terms, factorise y; asymptotes swap
"sketch"	asymptotes, intercepts	dashed asymptotes labelled; two branches, each through its intercepts W2
Paper 2, $f(x) > g(x)$	intersections and the asymptote	the GDC for the intersections; the asymptote splits the solution

In the exam 1 *command terms*

Asymptote: give its equation, $x = \ldots$ or $y = \ldots$. **Sketch**: both branches, the asymptotes dashed and labelled, the intercepts. **Write down**: read from the formula, no working needed.

V ON THE GDC: PAPER 2

SL 2.8 · NumWorks · every calculator move of this topic

1 See the asymptotes

Grapher → f(x) → Table → x close to 1.5

The table near the vertical asymptote and far away confirms both lines. **On paper:** the equations $x = 1.5$ and $y = 0.5$, from the formula.

GRAPH · TABLE

```
f(x)=(x+4)/(2x-3)
x=1.49 f=-274.5
x=1.51 f=275.5
x=1000 f=0.5027541
```

2 Where a line meets the hyperbola

Grapher → f(x), g(x) → Find → Intersection

Each branch can meet the line: look for an intersection on both sides of the asymptote. **On paper:** $x = -0.158$ or $x = 3.16$.

GRAPHER · INTERSECTION

```
f(x)=(x+4)/(2x-3) g(x)=x-1
x=-0.1583124 y=-1.158312
x=3.158312 y=2.158312
```

ANSWERS TO THE APPLY QUESTIONS

I.1 100

I.2 $x = -0.2$

I.3 $x \neq 0;\ y \neq 0$

I.4 $-\dfrac{3}{x}$

I.5 $x = 1$ or $x = -1$, on the line $y = x$

II.1 $x = 4, y = 1$

II.2 $x = \frac{1}{3}, y = 2$

II.3 $x = -1, y = -1$

II.4 $(-4, 0), (0, -8)$

II.5 $(2, 0), (0, -2)$

II.6 $x \neq 5;\ f(x) \neq 4$

II.7 $x \neq -6;\ f(x) \neq -2$

II.8 $\dfrac{x+4}{x-2}$

II.9 $k = 2$

II.10 $b = -6, a = -4$

II.11 $a = 2, b = -5$

II.12 $\dfrac{4x+3}{2-x}$

II.13 $x = -0.791$ or $x = 3.79$

II.14 $x = -0.414$ or $x = 2.41$

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