Inverse trigonometric functions

The equation $\sin x = \frac{1}{2}$ has infinitely many solutions, so sin has no inverse on $\mathbb{R}$. Restrict it to one interval where it is one-to-one and it has one: arcsin. The whole chapter is in the three restricted intervals, because they are the ranges of arcsin, arccos and arctan, and every answer must land in them.

I ARCSIN, ARCCOS AND ARCTAN

AHL 3.9 · domains and ranges, $\arcsin(\sin x)$, $\sin(\arcsin x)$

I.1 The three inverse functions

Definition 1 *the inverse functions*

$$\arcsin : [-1,1] \to \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right],$$

$$\arccos : [-1,1] \to [0,\pi],$$

$$\arctan : \mathbb{R} \to \left]-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right[.$$

$\arcsin x$ is the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x; likewise for the other two. They are also written $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$.

Example 1

Find the exact values of $\arccos\left(-\frac{\sqrt{3}}{2}\right)$, $\arctan\left(-\frac{1}{\sqrt{3}}\right)$ and $\arcsin\frac{\sqrt{2}}{2}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 arccos: the angle in $[0,\pi]$	$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$: $\frac{5\pi}{6}$
Step 2 arctan: the angle in $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$	$\tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$: $-\frac{\pi}{6}$
Step 3 arcsin: the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$\frac{\pi}{4}$

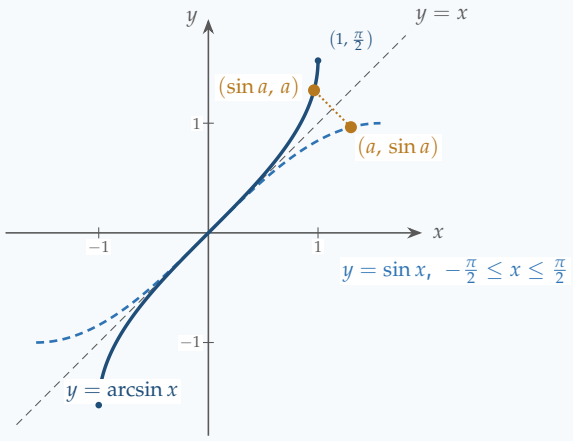


Figure 1 — $y = \sin x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and its reflection in $y = x$, the graph of $y = \arcsin x$: the point $(a, \sin a)$ becomes $(\sin a, a)$.

APPLY

1 $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$.

2 $\arccos\frac{\sqrt{2}}{2}$ and $\arctan\sqrt{3}$.

I.2 Compositions and symmetries

Property 1 *compositions and symmetries*

$\sin(\arcsin x) = x$ for every x in $[-1,1]$, but $\arcsin(\sin x) = x$ **only** for x in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. arcsin and arctan are odd: $\arcsin(-x) = -\arcsin x$; while $\arccos(-x) = \pi - \arccos x$. To find $\cos(\arcsin x)$, name the angle $\alpha = \arcsin x$ and use $\cos^2\alpha + \sin^2\alpha = 1$.

Example 2

Find the exact values of $\sin\left(\arccos\left(-\frac{3}{5}\right)\right)$ and $\arcsin\left(\sin\frac{2\pi}{3}\right)$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Name the angle	$\alpha = \arccos\left(-\frac{3}{5}\right),\ 0 \le \alpha \le \pi$
Step 2 Pythagoras, sign from the range	$\sin^2\alpha = 1 - \frac{9}{25} = \frac{16}{25};\ \sin\alpha \ge 0$ on $[0,\pi]$: $\frac{4}{5}$
Step 3 Same sine, angle in the range	$\sin\frac{2\pi}{3} = \sin\frac{\pi}{3}$, and $\frac{\pi}{3} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$: $\frac{\pi}{3}$

Watch out 1

$\arcsin\left(\sin\frac{5\pi}{6}\right) = \frac{\pi}{6}$, not $\frac{5\pi}{6}$: the answer must be in the range of arcsin. And $\sin^{-1}x$ is not $\frac{1}{\sin x}$, which is cosec x.

APPLY

3 $\cos(\arctan 2)$.

4 $\arccos\left(\cos\left(-\frac{\pi}{5}\right)\right)$.

II.1 The graphs

Property 2 *the graphs*

$y = \arcsin x$ rises from $\left(-1, -\frac{\pi}{2}\right)$ to $\left(1, \frac{\pi}{2}\right)$; $y = \arccos x$ falls from $(-1, \pi)$ to $(1, 0)$; $y = \arctan x$ rises on all of $\mathbb{R}$ between the horizontal asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$, which it never reaches.

Example 3

State the domain and the range of $y = \arccos x - \dfrac{\pi}{2}$, and three points of its graph.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Domain: that of arccos	$-1 \le x \le 1$
Step 2 Range: $[0, \pi]$ moved down	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
Step 3 Key points	$\left(-1, \frac{\pi}{2}\right)$, $(0, 0)$, $\left(1, -\frac{\pi}{2}\right)$: decreasing

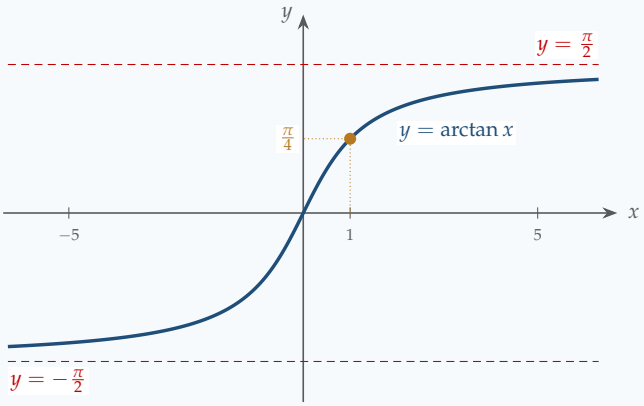


Figure 2 — $y = \arctan x$ approaches $y = \frac{\pi}{2}$ as $x \to \infty$ and $y = -\frac{\pi}{2}$ as $x \to -\infty$; $\arctan 1 = \frac{\pi}{4}$.

APPLY

1 The range of $y = 3\arctan x$.

2 The asymptotes of $y = \arctan x + 1$.

II.2 Domains and inverses

Property 3 *domain and inverse*

$\arcsin(u)$ and $\arccos(u)$ exist only for $-1 \le u \le 1$: the domain of $\arccos(2x-1)$ is $0 \le x \le 1$. A function $f(x) = a\sin(bx) + d$ on an interval where bx stays in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is one-to-one, and

$$f^{-1}(x) = \frac{1}{b}\arcsin\frac{x-d}{a},$$

with the range of f as its domain.

Example 4

$f(x) = 4\sin\frac{x}{2} - 1$, for $-\pi \le x \le \pi$. Find the range of f, then $f^{-1}(x)$ and its domain.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Range: f is increasing, values at the ends	$f(-\pi) = -5,\ f(\pi) = 3$: $-5 \le f(x) \le 3$
Step 2 Swap, then divide	$x = 4\sin\frac{y}{2} - 1$: $\sin\frac{y}{2} = \frac{x+1}{4}$
Step 3 arcsin, then multiply by 2	$f^{-1}(x) = 2\arcsin\dfrac{x+1}{4}$
Step 4 Domain: the range of f	$-5 \le x \le 3$

Watch out 2

Divide **before** taking arcsin: from $x = 3\sin 2y + 1$, first $\sin 2y = \frac{x-1}{3}$, then $2y = \arcsin\frac{x-1}{3}$. The domain of f^{-1} is the range of f, not the domain of f.

APPLY

3 The domain of $\arccos\left(\frac{x}{3} + 1\right)$.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A Exact values and compositions

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 7]

Find the exact value of

(a) $\arcsin\left(-\frac{1}{2}\right)$; [1]

(b) $\arccos\left(-\frac{\sqrt{2}}{2}\right)$; [1]

(c) $\arcsin\left(\sin\frac{4\pi}{5}\right)$; [2]

(d) $\tan\left(\arcsin\frac{1}{3}\right)$. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The angle in the range	$-\frac{\pi}{6}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The angle in the range	$\frac{3\pi}{4}$ *In* $[0,\pi]$.	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Same sine, angle in the range	$\sin\frac{4\pi}{5}=\sin\frac{\pi}{5}$ $\pi-\frac{4\pi}{5}$, *which is in* $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.	M1
Step 2 The value	$\frac{\pi}{5}$	A1

WORKING (d)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Name the angle, Pythagoras	$\alpha=\arcsin\frac{1}{3},\ \cos^2\alpha=1-\frac{1}{9}=\frac{8}{9}$	M1
Step 2 Sign from the range	$\cos\alpha=\frac{2\sqrt{2}}{3}$, positive since $-\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}$	R1
Step 3 The value	$\tan\alpha=\frac{1/3}{2\sqrt{2}/3}=\frac{\sqrt{2}}{4}$ *Or* $\frac{1}{2\sqrt{2}}$.	A1

B The inverse of a restricted sine

EXAM-STYLE QUESTION 2 Paper 1 · no calculator · [Maximum mark: 7]

Let $f(x) = 3\sin(2x) + 1$, for $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$.

(a) Find the range of f. [2]

(b) Find an expression for $f^{-1}(x)$, and state its domain. [3]

(c) Solve the equation $f^{-1}(x) = \frac{\pi}{12}$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Values at the ends	$f\left(-\frac{\pi}{4}\right) = -2,\ f\left(\frac{\pi}{4}\right) = 4$ *Either value.*	A1
Step 2 The range	$-2 \leq f(x) \leq 4$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Swap, then divide	$x = 3\sin(2y) + 1,\ \sin(2y) = \frac{x-1}{3}$ *Interchanging x and y, at any stage.*	M1
Step 2 arcsin, then ÷2	$f^{-1}(x) = \frac{1}{2}\arcsin\left(\frac{x-1}{3}\right)$	A1
Step 3 Domain: the range of f	$-2 \leq x \leq 4$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Apply f to both sides	$\arcsin\left(\frac{x-1}{3}\right) = \frac{\pi}{6},\ \frac{x-1}{3} = \frac{1}{2}$ *Or $x = f\left(\frac{\pi}{12}\right)$.*	M1
Step 2 Solve	$x = \frac{5}{2}$	A1

C On the GDC (Paper 2)

EXAM-STYLE QUESTION 3 Paper 2 · calculator · [Maximum mark: 6]

Let $f(x) = \arccos\left(\frac{x}{2}\right) - \arctan x$, for $-2 \leq x \leq 2$.

(a) Find the exact value of $f\left(\sqrt{3}\right)$. [1]

(b) Solve the equation $f(x) = 1$. [2]

(c) Find the range of f. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Exact values	$\frac{\pi}{6} - \frac{\pi}{3} = -\frac{\pi}{6}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The equation on the GDC	$\arccos\left(\frac{x}{2}\right) - \arctan x = 1$ *For example, the graphs of $y = f(x)$ and $y = 1$ intersected.*	(M1)
Step 2 Solve	$x = 0.392$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Decreasing	f is decreasing, since $\arccos\frac{x}{2}$ and $-\arctan x$ both decrease	R1
Step 2 Values at the ends	$f(2) = -\arctan 2 = -1.11$, $f(-2) = \pi + \arctan 2 = 4.25$	A1
Step 3 The range	$-1.11 \leq f(x) \leq 4.25$	A1

IV WHICH METHOD? READ THE QUESTION

AHL 3.9 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"exact value" of arcsin, arccos, arctan	the range of the inverse	the one angle in that range D1
$\arcsin(\sin a)$ *with* a *outside* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	same sine, angle in the range	$\pi - a$, or $a - 2\pi$... W1
$\cos(\arcsin x)$, $\tan(\arccos x)$...	name the angle α	$\cos^2\alpha + \sin^2\alpha = 1$, sign from the range P1
$\arccos(-x)$	$\pi - \arccos x$	an angle in $\left[\frac{\pi}{2}, \pi\right]$ P1
the domain of $\arcsin(ax+b)$	$-1 \le ax + b \le 1$	solve the double inequality P3
"find f^{-1}*" for* $a\sin(bx)+d$	divide before arcsin	$\frac{1}{b}\arcsin\frac{x-d}{a}$, domain = range of f P3
the asymptotes of arctan	$\pm\frac{\pi}{2}$, never reached	$y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ P2

In the exam 1 *command terms*

Find the exact value: an angle in radians, as a multiple of π or a surd. **State its domain**: an interval, with its ends included or not. **Find the range**: the smallest and largest values of f, with the reason when f is monotonic.

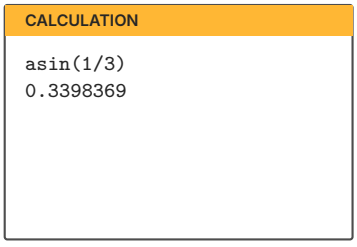
V ON THE GDC: PAPER 2

AHL 3.9 · NumWorks · every calculator move of this topic

1 Inverse ratios in radians

Settings → Radians → Calculation → shift sin → asin(1/3)

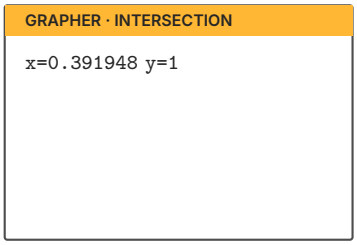
arcsin, arccos and arctan are the second functions of sin, cos and tan. Check that the angle unit is radians first. **On paper:** the angle to three significant figures.



2 An equation with inverse ratios

Grapher → f(x)=acos(x/2)-atan(x) → g(x)=1 → Calculate → Intersection

Draw both sides on the domain of the question and find their intersection. **On paper:** the equation written, then x to three significant figures.



ANSWERS TO THE APPLY QUESTIONS

I.1 $-\frac{\pi}{3}$

I.2 $\frac{\pi}{4}$; $\frac{\pi}{3}$

I.3 $\frac{1}{\sqrt{5}}$

I.4 $\frac{\pi}{5}$

II.1 $\left]-\frac{3\pi}{2}, \frac{3\pi}{2}\right[$

II.2 $y = 1 \pm \frac{\pi}{2}$

II.3 $-6 \leq x \leq 0$

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