Reciprocal trigonometric functions

Three new names for three old fractions: $\sec x$, $\operatorname{cosec} x$ and $\cot x$ are one over $\cos x$, $\sin x$ and $\tan x$. Each takes its sign from its partner, and each has an asymptote wherever its partner is zero. Two new identities follow from $\cos^2 x + \sin^2 x = 1$, and they turn a mixed equation into a quadratic in one ratio.

I SECANT, COSECANT AND COTANGENT

AHL 3.9 · $\sec x = \frac{1}{\cos x}$, $\operatorname{cosec} x = \frac{1}{\sin x}$, $\cot x = \frac{1}{\tan x}$

I.1 The reciprocal ratios

Definition 1 *the reciprocal ratios*

$$\sec x = \frac{1}{\cos x}, \qquad \operatorname{cosec} x = \frac{1}{\sin x}, \qquad \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}.$$

Each is undefined where its denominator is 0: $\sec x$ at $x = \frac{\pi}{2} + k\pi$, $\operatorname{cosec} x$ and $\cot x$ at $x = k\pi$.

Example 1

Find the exact values of $\sec \frac{5\pi}{6}$, $\operatorname{cosec}\left(-\frac{\pi}{6}\right)$ and $\cot \frac{2\pi}{3}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 sec: one over the exact cos	$\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$: $\sec \frac{5\pi}{6} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$
Step 2 cosec: one over the exact sin	$\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$: $\operatorname{cosec}\left(-\frac{\pi}{6}\right) = -2$
Step 3 cot: one over the exact tan	$\tan \frac{2\pi}{3} = -\sqrt{3}$: $\cot \frac{2\pi}{3} = -\frac{\sqrt{3}}{3}$

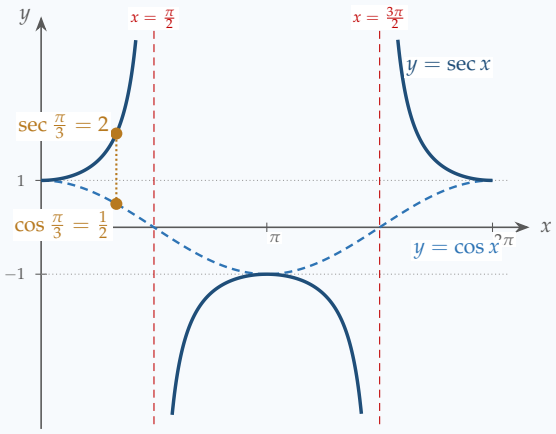


Figure 1 — $y = \sec x$ (solid) and $y = \cos x$ (dashed): where $\cos x = \pm 1$ they touch; where $\cos x = 0$, $\sec x$ has a vertical asymptote. Since $|\cos x| \leq 1$, $|\sec x| \geq 1$.

APPLY

1 Where is $\operatorname{cosec} x$ undefined, for $0 \leq x \leq 2\pi$?

2 $\cot \frac{\pi}{3}$, exactly.

I.2 Signs, ranges and graphs

Property 1 *signs, values and graphs*

$\sec x$ has the sign of $\cos x$, $\operatorname{cosec} x$ and $\cot x$ the sign of $\sin x$ and $\tan x$. Exact values come from the usual ones: $\sec \frac{\pi}{3} = 2$, $\operatorname{cosec} \frac{\pi}{6} = 2$, $\cot \frac{\pi}{4} = 1$. sec and cosec have period 2π and range $y \leq -1$ or $y \geq 1$; cot has period π and range $\mathbb{R}$.

Example 2

$\cos\theta = -\dfrac{3}{5}$ and $\pi < \theta < \dfrac{3\pi}{2}$. Find $\sec\theta$, $\operatorname{cosec}\theta$ and $\cot\theta$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 sec: the reciprocal	$\sec\theta = -\frac{5}{3}$
Step 2 $\sin\theta$, sign from the quadrant	third quadrant: $\sin\theta = -\frac{4}{5}$
Step 3 cosec and cot	$\operatorname{cosec}\theta = -\frac{5}{4}$, $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \frac{3}{4}$

Watch out 1

$\sec x$ is **not** $\cos^{-1} x$: the index -1 on a function name means the inverse function, $\arccos x$. One over $\cos x$ is written $\sec x$ or $(\cos x)^{-1}$.

APPLY

3 The range of $y = 3\sec x$.

4 The sign of $\cot 4$ (4 radians).

II THE PYTHAGOREAN IDENTITIES

AHL 3.9 · $1+\tan^2 x = \sec^2 x$, $1+\cot^2 x = \operatorname{cosec}^2 x$

II.1 Two identities from one

Property 2 *two identities from one*

Divide $\cos^2 x + \sin^2 x = 1$ by $\cos^2 x$, then by $\sin^2 x$:

$$1 + \tan^2 x = \sec^2 x,$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x.$$

From one ratio and the quadrant, they give the reciprocal of the other without an angle.

Example 3

$\tan\theta = 2\sqrt{2}$ and $\pi < \theta < \dfrac{3\pi}{2}$. Find the exact values of $\sec\theta$ and $\operatorname{cosec}\theta$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 $1+\tan^2\theta = \sec^2\theta$	$\sec^2\theta = 1 + 8 = 9$
Step 2 Sign from the quadrant	third quadrant, $\cos\theta < 0$: $\sec\theta = -3$
Step 3 $1+\cot^2\theta = \operatorname{cosec}^2\theta$	$\cot\theta = \frac{1}{2\sqrt{2}}$: $\operatorname{cosec}^2\theta = 1 + \frac{1}{8} = \frac{9}{8}$
Step 4 Sign from the quadrant	$\sin\theta < 0$: $\operatorname{cosec}\theta = -\frac{3}{2\sqrt{2}} = -\frac{3\sqrt{2}}{4}$

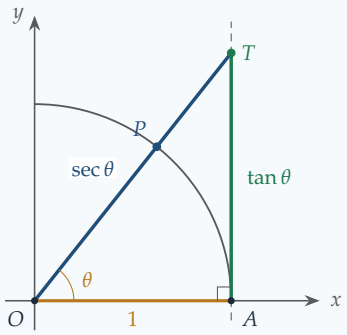


Figure 2 — On the unit circle, the tangent at $A(1,0)$ meets OP at T: $AT = \tan\theta$ and $OT = \sec\theta$. Pythagoras in OAT is $1 + \tan^2\theta = \sec^2\theta$.

Watch out 2

The square root has two signs: $\sec\theta = \pm\sqrt{1+\tan^2\theta}$, and the quadrant chooses. In the second quadrant $\cos\theta < 0$, so $\sec\theta < 0$ even though $\tan\theta$ is given.

APPLY

1 $\cot\theta = -\frac{\sqrt{5}}{2}$, $\frac{\pi}{2} < \theta < \pi$: $\operatorname{cosec}\theta$.

2 $\sec\theta = 3$, $0 < \theta < \frac{\pi}{2}$: $\tan\theta$.

II.2 A quadratic in one ratio

Property 3 *a quadratic in one ratio*

An equation that mixes $\tan^2 x$ and $\sec x$ (or $\cot^2 x$ and $\operatorname{cosec} x$) becomes a quadratic in $\sec x$ (or $\operatorname{cosec} x$) once the square is replaced. Solve it, reject any root with $|\sec x| < 1$, then go back to $\cos x$ (or $\sin x$).

Example 4

Solve $2\tan^2 x - 5\sec x + 4 = 0$, for $0 \leq x < 2\pi$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Replace $\tan^2 x$ by $\sec^2 x - 1$	$2\sec^2 x - 5\sec x + 2 = 0$
Step 2 Factorise	$(2\sec x - 1)(\sec x - 2) = 0$
Step 3 Reject $\|\sec x\| < 1$	$\sec x = \frac{1}{2}$ is impossible: $\sec x = 2$
Step 4 Back to $\cos x$, solve	$\cos x = \frac{1}{2}$: $x = \frac{\pi}{3}$, $x = \frac{5\pi}{3}$

APPLY

3 Solve $\cot^2 x = \operatorname{cosec} x + 1$, for $0 < x < 2\pi$.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A Exact values

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 5]

Find the exact value of

(a) $\sec \frac{2\pi}{3}$; [2]

(b) $\cot \frac{7\pi}{6}$; [2]

(c) $\operatorname{cosec}^2 \frac{\pi}{4}$. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The exact cos	$\cos \frac{2\pi}{3} = -\frac{1}{2}$	M1
Step 2 Its reciprocal	$\sec \frac{2\pi}{3} = -2$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $\cot = \frac{\cos}{\sin}$	$\frac{\cos \frac{7\pi}{6}}{\sin \frac{7\pi}{6}} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$	M1
	Or $\tan \frac{7\pi}{6} = \frac{1}{\sqrt{3}}$, *then the reciprocal.*	
Step 2 Simplify	$\sqrt{3}$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Square of the reciprocal	2	A1
	$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$, *so* $\operatorname{cosec} \frac{\pi}{4} = \sqrt{2}$.	

EXAM-STYLE QUESTION 2

Paper 1 · no calculator · [Maximum mark: 7]

It is given that $\tan\theta = -\frac{2\sqrt{6}}{5}$, where $\frac{\pi}{2} < \theta < \pi$.

(a) Find the exact value of $\sec\theta$. [3]

(b) Find the exact value of $\operatorname{cosec}\theta$. [2]

(c) Hence find the exact value of $\sin 2\theta$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $1 + \tan^2\theta = \sec^2\theta$	$\sec^2\theta = 1 + \frac{24}{25} = \frac{49}{25}$	M1
Step 2 Sign from the quadrant	$\sec\theta < 0$ because $\cos\theta < 0$ for $\frac{\pi}{2} < \theta < \pi$	R1
Step 3 The value	$\sec\theta = -\frac{7}{5}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $1 + \cot^2\theta = \operatorname{cosec}^2\theta$	$\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \frac{25}{24} = \frac{49}{24}$	M1
	Or $\sin\theta = \tan\theta\cos\theta = \frac{2\sqrt{6}}{7}$.	
Step 2 The positive root	$\operatorname{cosec}\theta = \frac{7}{2\sqrt{6}} = \frac{7\sqrt{6}}{12}$	A1
	Positive: $\sin\theta > 0$ *in the second quadrant.*	

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $\sin\theta$ and $\cos\theta$	$\sin\theta = \frac{2\sqrt{6}}{7}$, $\cos\theta = -\frac{5}{7}$, $2 \times \frac{2\sqrt{6}}{7} \times \left(-\frac{5}{7}\right)$	M1
Step 2 Double angle	$\sin 2\theta = -\frac{20\sqrt{6}}{49}$	A1

C An equation on the GDC (Paper 2)

EXAM-STYLE QUESTION 3 Paper 2 · calculator · [Maximum mark: 7]

Consider the equation $2\cot^2 x + 3\operatorname{cosec} x = 4$, for $0 < x < 2\pi$, $x \neq \pi$.

(a) Show that the equation can be written as $2\operatorname{cosec}^2 x + 3\operatorname{cosec} x - 6 = 0$. [2]

(b) Hence solve the equation. [5]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Replace $\cot^2 x$	$2\left(\operatorname{cosec}^2 x - 1\right) + 3\operatorname{cosec} x = 4$ *Using* $\cot^2 x = \operatorname{cosec}^2 x - 1$.	M1
Step 2 Collect	$2\operatorname{cosec}^2 x + 3\operatorname{cosec} x - 6 = 0$ *AG.*	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Solve the quadratic	$\operatorname{cosec} x = 1.13746$ or $\operatorname{cosec} x = -2.63746$ $\frac{-3 \pm \sqrt{57}}{4}$.	(M1)A1
Step 2 Back to $\sin x$	$\sin x = 0.879153$ or $\sin x = -0.379153$ *Taking reciprocals; both values have* $\lvert\sin x\rvert < 1$*, so both give solutions.*	M1
Step 3 First pair of solutions	$x = 1.07,\ x = 2.07$	A1
Step 4 Second pair	$x = 3.53,\ x = 5.89$	A1

IV WHICH METHOD? READ THE QUESTION

AHL 3.9 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"exact value" of sec, cosec *or* cot	one over cos, sin, tan	the exact cos, sin or tan, then its reciprocal D1
the sign of $\sec\theta$, $\operatorname{cosec}\theta$, $\cot\theta$	the sign of its partner	the quadrant decides P1
where is $\sec x$ *(or* $\operatorname{cosec} x$*) undefined*	the partner is 0	vertical asymptotes $x = \frac{\pi}{2} + k\pi$ (or $x = k\pi$) D1
$\tan\theta$ *given,* $\sec\theta$ *wanted*	$1 + \tan^2\theta = \sec^2\theta$	square root, sign from the quadrant P2
$\cot\theta$ *given,* $\operatorname{cosec}\theta$ *wanted*	$1 + \cot^2\theta = \operatorname{cosec}^2\theta$	square root, sign of $\sin\theta$ P2
an equation with $\tan^2 x$ *and* $\sec x$	a quadratic in $\sec x$	replace $\tan^2 x$ by $\sec^2 x - 1$, factorise P3
a root $\sec x = k$ *with* $\lvert k\rvert < 1$	$\lvert\sec x\rvert \geq 1$	reject it, with the reason P1

In the exam 1 *command terms*

Find the exact value: a surd or a fraction, no decimal. **Hence**: use the answer just found. **Show that**: start from one side, use an identity, and end exactly on the other.

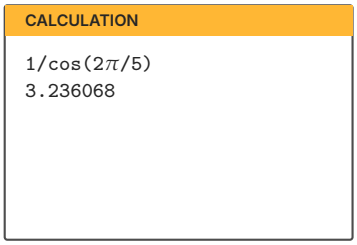
V ON THE GDC: PAPER 2

AHL 3.9 · NumWorks · every calculator move of this topic

1 Reciprocal ratios on the GDC

Calculation → 1/cos(2π/5)

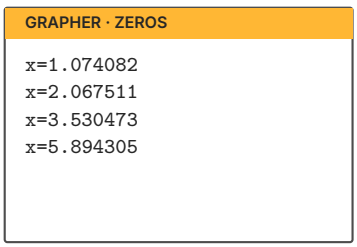
The NumWorks has no sec key: type one over cos, sin or tan, in radians. **On paper:** the reciprocal written out, then the value to three significant figures.



2 Every solution in the interval

Grapher → f(x)=2/tan(x)^2+3/sin(x)-4 → Axes: 0 to 2π → Calculate → Zeros

Draw the whole equation as one function, with the window on the interval of the question, and read every zero. **On paper:** the quadratic in $\operatorname{cosec} x$ first, then all four values.



ANSWERS TO THE APPLY QUESTIONS

I.1 $x = 0, \pi, 2\pi$

I.2 $\frac{\sqrt{3}}{3}$

I.3 $y \leq -3$ or $y \geq 3$

I.4 third quadrant, $\tan 4 > 0$: positive

II.1 $\frac{3}{2}$

II.2 $2\sqrt{2}$

II.3 $(\operatorname{cosec} x - 2)(\operatorname{cosec} x + 1) = 0$: $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

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