Solids and 3D geometry

Space adds one coordinate, z, and one difficulty: drawing. Every question of this topic comes back to a flat right-angled triangle hidden in the solid. Find it, draw it on its own, and the distance, the volume or the angle follows.

I POINTS AND SOLIDS

SL 3.1 · distance and midpoint in 3D, volume and surface area

I.1 Distance and midpoint in three dimensions

Definition 1 *distance and midpoint in three dimensions*

For $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$: $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ and $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

Example 1

Find the distance between $P(2, -1, 4)$ and $Q(5, 3, -8)$, and the midpoint of $[PQ]$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 The three differences	$3,\ 4,\ -12$
Step 2 Square, add, square root	$PQ = \sqrt{9 + 16 + 144} = \sqrt{169} = 13$
Step 3 Midpoint: the mean of each coordinate	$(3.5,\ 1,\ -2)$

APPLY

1 The distance from $(0, 0, 0)$ to $(2, 6, 9)$.

2 The distance from $(4, -3, 1)$ to $(2, 1, 5)$.

3 The midpoint of $(-2, 5, 0)$ and $(6, 1, 4)$.

I.2 Volume of a pyramid and of a cone

Property 1 *volumes and surface areas*

Right pyramid $V = \frac{1}{3}Ah$ (A the base area). Right cone $V = \frac{1}{3}\pi r^2 h$, curved surface $\pi r l$, slant height $l = \sqrt{r^2 + h^2}$. Sphere $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$; hemisphere $V = \frac{2}{3}\pi r^3$, curved surface $2\pi r^2$.

Example 2

Find the volume of a pyramid with a rectangular base 6 cm by 4 cm and height 9 cm, and of a cone with radius 3 cm and height 7 cm.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Pyramid: base area, then $\frac{1}{3}Ah$	$A = 24,\ V = \frac{1}{3} \times 24 \times 9 = 72$ cm^3
Step 2 Cone: $\frac{1}{3}\pi r^2 h$	$\frac{1}{3}\pi \times 9 \times 7 = 21\pi$
Step 3 Round at the end	66.0 cm^3

APPLY

4 A pyramid with a square base of side 9 cm and height 10 cm.

5 A cone with radius 10 cm and height 6 cm.

I.3 Slant height and curved surface

Watch out 1

The height h is vertical, from the apex to the centre of the base; the slant height l runs along the surface. V uses h; the curved surface uses l.

Example 3

A cone has radius 8 cm and height 15 cm. Find its slant height and its curved surface area.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Right-angled triangle h, r, l	$l = \sqrt{8^2 + 15^2} = 17$ cm
Step 2 Curved surface: $\pi r l$	$\pi \times 8 \times 17 = 136\pi$
Step 3 Round at the end	427 cm^2

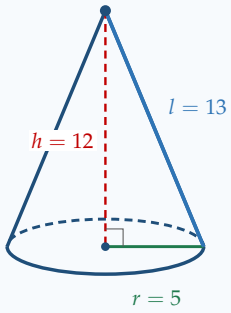


Figure 1 — Exam-style question 2's cone, radius 5 and height 12: h, r and l make a right-angled triangle, $l = 13$.

APPLY

6 $r = 7$ cm, $h = 24$ cm: slant height and curved surface.

7 $r = 9$ cm, slant height 15 cm: the height.

I.4 Spheres and hemispheres

Example 4

A sphere has radius 4.5 cm. Find its volume and its surface area.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Volume: $\frac{4}{3}\pi r^3$	$\frac{4}{3}\pi \times 4.5^3 = 121.5\pi = 382$ cm^3
Step 2 Surface: $4\pi r^2$	$4\pi \times 4.5^2 = 81\pi = 254$ cm^2

APPLY

8 A hemisphere of radius 9 cm: its volume.

9 A sphere has surface area 100π cm^2: its radius.

I.5 Combined solids

Example 5

A solid is a cylinder of radius 3 cm and height 10 cm with a hemisphere on top. Find its volume and its total surface area.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Volume: add the parts	$\pi \times 9 \times 10 + \frac{2}{3}\pi \times 27 = 90\pi + 18\pi = 108\pi = 339$ cm^3
Step 2 List the outside surfaces No flat face where the two parts meet.	the base, the side of the cylinder, the curved hemisphere
Step 3 Add them	$9\pi + 60\pi + 18\pi = 87\pi = 273$ cm^2

APPLY

10 A cone ($r = 4$ cm, $h = 3$ cm) on a cylinder ($r = 4$ cm, $h = 6$ cm): the volume.

II ANGLES IN SOLIDS

SL 3.1 · the angle between two lines, between a line and a plane

II.1 The angle between a line and a plane

Definition 2 *the angle between a line and a plane*

Drop a perpendicular from a point of the line to the plane: its foot gives the **projection** of the line. The angle with the plane is the angle between the line and its projection.

Property 2 *the right-angled triangle*

The line, its projection and the perpendicular form a right-angled triangle: $\tan\theta = \dfrac{\text{perpendicular}}{\text{projection}}$.

Watch out 2

The angle with the base is not the angle with an edge of the base. Find the projection first.

Example 6

A cuboid $ABCDEFGH$ has $AB = 4$ cm, $BC = 3$ cm and $AE = 12$ cm. Find the angle between the diagonal AG and the base $ABCD$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Projection of AG on the base	$AC = \sqrt{4^2 + 3^2} = 5$
Step 2 The perpendicular	$CG = 12$
Step 3 $\tan\theta = \dfrac{\text{perpendicular}}{\text{projection}}$	$\tan\theta = \dfrac{12}{5}$, $\theta = 67.4°$

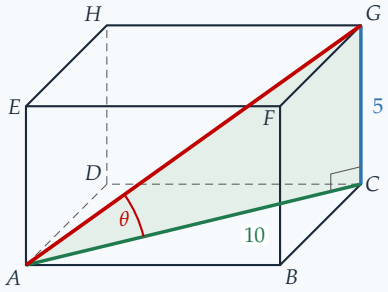


Figure 2 — Exam-style question 3's cuboid, $8 \times 6 \times 5$: AG projects onto AC; right angle at C, $\tan\theta = \frac{5}{10}$.

APPLY

1 A cuboid with base 5 cm by 12 cm and height 6 cm: the angle between a long diagonal and the base.

2 A square-based pyramid, side 8 cm, apex 6 cm above the centre: the angle between VA and the base.

II.2 The angle between two lines

Example 7

In the cuboid of Example 6 ($AB = 4$, $BC = 3$, $AE = 12$), find the angle $F\hat{A}C$ between the face diagonals AF and AC.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Three sides of triangle AFC, by Pythagoras	$AF = \sqrt{160},\ AC = 5,\ FC = \sqrt{153}$
Step 2 Cosine rule	$\cos \hat{A} = \dfrac{160 + 25 - 153}{2 \times \sqrt{160} \times 5} = 0.25298\ldots$
Step 3 Answer	$\hat{A} = 75.3°$

APPLY

3 A cube of side 2: the angle between the face diagonals AC and AF.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A Points and spheres

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 6]

The points $A(1, -2, 3)$ and $B(5, 2, 1)$ are the ends of a diameter of a sphere.

(a) Find the distance AB. [2]

(b) Find the coordinates of the centre of the sphere. [2]

(c) Find the volume of the sphere. Give your answer in terms of π. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Square the differences, add, square root	$\sqrt{4^2 + 4^2 + (-2)^2}$	M1
Step 2 Compute	$AB = 6$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Midpoint: the mean of each coordinate	$\left(\frac{1+5}{2}, \frac{-2+2}{2}, \frac{3+1}{2}\right)$	(M1)
	The centre is the midpoint of the diameter.	
Step 2 Write the point	$(3, 0, 2)$	A1

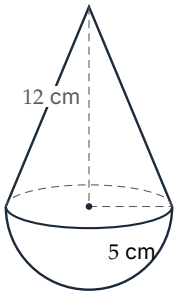
WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Radius: half of AB	$r = 3$	(A1)
Step 2 $V = \frac{4}{3}\pi r^3$	$V = \frac{4}{3}\pi \times 3^3 = 36\pi$	A1

B A combined solid (Paper 2)

EXAM-STYLE QUESTION 2 Paper 2 · calculator · [Maximum mark: 7]

A solid is made of a right cone on top of a hemisphere, as shown. The cone and the hemisphere both have radius 5 cm; the height of the cone is 12 cm.



(a) Find the slant height of the cone. [2]
(b) Find the volume of the solid. [3]
(c) Find the total surface area of the solid. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Right-angled triangle h, r, l	$l^2 = 5^2 + 12^2$	(M1)
Step 2 Compute	$l = 13$ cm	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Volume of the cone	cone: $\frac{1}{3}\pi \times 5^2 \times 12 = 100\pi$	(A1)
Step 2 Volume of the hemisphere	hemisphere: $\frac{2}{3}\pi \times 5^3 = \frac{250\pi}{3}$	(A1)
Step 3 Add	$V = 576$ cm^3 $(575.958\ldots)$	A1

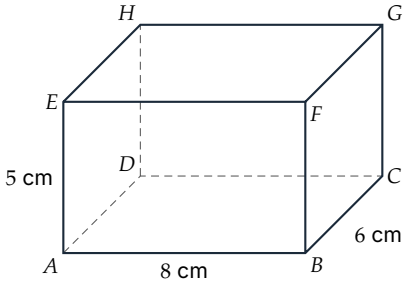
WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Outside surfaces only: cone and hemisphere	$\pi \times 5 \times 13 + 2\pi \times 5^2$	(M1)
	The curved surfaces only: the flat faces are inside the solid.	
Step 2 Compute	361 cm^2 $(115\pi = 361.283\ldots)$	A1

C Angles in solids (Paper 2)

EXAM-STYLE QUESTION 3 Paper 2 · calculator · [Maximum mark: 6]

The diagram shows a cuboid $ABCDEFGH$ with $AB = 8$ cm, $BC = 6$ cm and $AE = 5$ cm.



(a) Find AC. [2]

(b) Find AG. [2]

(c) Find the angle between AG and the base $ABCD$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Pythagoras in the base	$AC^2 = 8^2 + 6^2$	(M1)
Step 2 Compute	$AC = 10$ cm	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Pythagoras in triangle ACG	$AG^2 = 10^2 + 5^2$ *Triangle ACG, right-angled at C.*	(M1)
Step 2 Compute	$AG = 11.2$ cm $(\sqrt{125} = 11.1803\ldots)$	A1

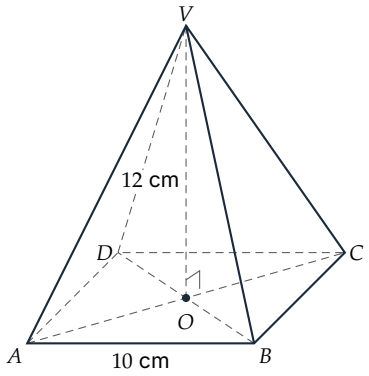
WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $\tan\theta = \dfrac{CG}{AC}$	$\tan\theta = \dfrac{5}{10}$ *Or* $\sin\theta = \dfrac{5}{11.18\ldots}$.	(M1)
Step 2 Compute	$\theta = 26.6°$ $(26.5650\ldots)$	A1

EXAM-STYLE QUESTION 4

Paper 2 · calculator · [Maximum mark: 6]

A right pyramid $VABCD$ has a square base $ABCD$ of side 10 cm. Its apex V is 12 cm vertically above the centre O of the base.



(a) Find the volume of the pyramid. [2]

(b) Find the height of the triangular face VAB, from V to the midpoint of AB. [2]

(c) Find the angle between the edge VA and the base $ABCD$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $V = \frac{1}{3}Ah$	$\frac{1}{3} \times 10^2 \times 12$	(M1)
Step 2 Compute	$V = 400 \text{ cm}^3$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Right-angled triangle: height and half the side	$\sqrt{12^2 + 5^2}$	(M1)
	From O to the midpoint of AB is half a side, 5.	
Step 2 Compute	13 cm	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Projection OA, then $\tan\theta$	$OA = 5\sqrt{2}$, $\tan\theta = \dfrac{12}{5\sqrt{2}}$	(M1)
	OA is half the diagonal of the base: the projection of VA.	
Step 2 Compute	$\theta = 59.5°$ $(59.4910\ldots)$	A1

IV WHICH METHOD? READ THE QUESTION

3.1 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
two points with three coordinates	$AB = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$	the differences, squared, added, square root D1
"centre", "midpoint"	the mean of each coordinate	$\left(\frac{x_1+x_2}{2}, \ldots\right)$ D1
a cone: radius, height, slant	$l^2 = r^2 + h^2$	V with h, curved surface with l P1
"the solid is made of"	add the volumes	surface: only the outside faces P1
"angle between the line and the plane"	the projection on the plane	the right-angled triangle, then $\tan\theta$ P2
angle between two lines in a solid	three sides of a triangle	Pythagoras for the sides, then the cosine rule P2
"in terms of π"	exact answer	keep π; no decimal

In the exam 1 *command terms*

Find: show the formula with the numbers in it, then the answer to 3 s.f. with units. **In terms of π**: an exact answer, for example 36π. **Show that**: every step, ending on the given result.

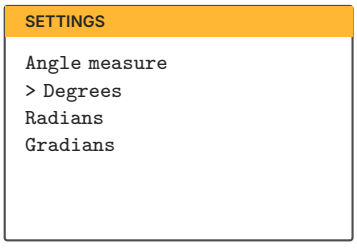
V ON THE GDC: PAPER 2

SL 3.1 · NumWorks · every calculator move of this topic

1 Degrees, before any angle

Settings → Angle measure → Degrees

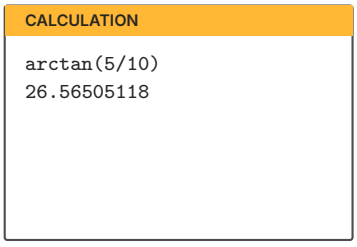
An angle in a solid is given in degrees: check the banner says deg before you start. **On paper:** the angle to 1 decimal place, with °.



2 An angle from its tangent

Calculation → shift → tan → 5/10 → EXE

shift then tan gives arctan. Type the exact ratio, not a rounded side. **On paper:** "$\tan\theta = \frac{5}{10}$, $\theta = 26.6°$".



3 A volume with π

Calculation → (1/3)×π×5^2×12 → EXE

Keep π on the calculator until the last step. **On paper:** the formula with its numbers, then 314 cm^3 — or 100π when the question says "in terms of π".

CALCULATION

```
(1/3)*π*5^2*12
314.1592654
```

ANSWERS TO THE APPLY QUESTIONS

I.1 11

I.2 6

I.3 $(2, 3, 2)$

I.4 270 cm^3

I.5 $200\pi = 628$ cm^3

I.6 25 cm; $175\pi = 550$ cm^2

I.7 12 cm

I.8 $486\pi = 1530$ cm^3

I.9 5 cm

I.10 $112\pi = 352$ cm^3

II.1 $24.8°$

II.2 $46.7°$

II.3 $60°$ (an equilateral triangle)

Written and verified by a practising IB mathematics teacher, produced with AI assistance; every result recomputed by software, every figure computed from its function. Not affiliated with or endorsed by the International Baccalaureate Organization. © 2026 Hence