The unit circle and identities

SL AA · Geometry and trigonometry · 3.5 · 3.6

A right-angled triangle only knows angles below 90°. The unit circle gives a sine and a cosine to every angle: they are the coordinates of a point turning round a circle of radius 1. Signs, exact values and identities all come from that one picture.

I THE UNIT CIRCLE

SL 3.5 · $\cos\theta$ and $\sin\theta$ on the unit circle, exact values, the ambiguous case

I.1 The point $(\cos\theta, \sin\theta)$

Definition 1 *cosine and sine of any angle*

Turn anticlockwise from the positive x-axis by θ on the circle of radius 1 centred at O: the point reached is $P(\cos\theta,\ \sin\theta)$, and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $(\cos\theta \neq 0)$, the gradient of OP.

Example 1

Find the exact coordinates of the point P of the unit circle at $\theta = \dfrac{7\pi}{4}$, and $\tan\dfrac{7\pi}{4}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Reference angle	$2\pi - \dfrac{7\pi}{4} = \dfrac{\pi}{4}$
Step 2 Signs in the quadrant	fourth quadrant: cos >0, sin <0
Step 3 Coordinates	$P\left(\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$
Step 4 tan $=$ sin / cos	$\tan\dfrac{7\pi}{4} = -1$

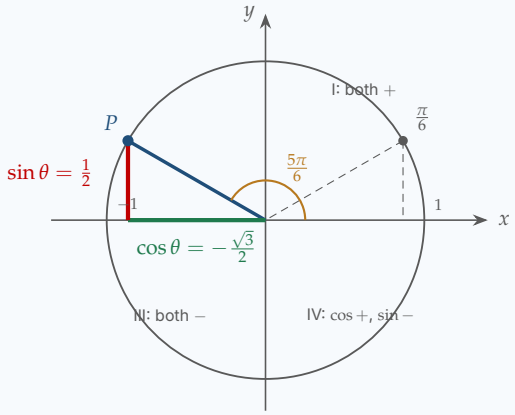


Figure 1 — Exam-style question 1: $\theta = \frac{5\pi}{6}$, $P\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, the reflection of the point at $\frac{\pi}{6}$ in the y-axis.

APPLY

1 The point at $\theta = \frac{5\pi}{4}$.

2 The point at $\theta = \frac{11\pi}{6}$.

3 The point at $\theta = \frac{3\pi}{2}$.

I.2 Exact values

Property 1 *exact values and symmetries*

$\sin\frac{\pi}{6} = \frac{1}{2}$, $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, cosines in the reverse order; $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, $\tan\frac{\pi}{4} = 1$, $\tan\frac{\pi}{3} = \sqrt{3}$. Symmetries: $\sin(\pi - \theta) = \sin\theta$, $\cos(\pi - \theta) = -\cos\theta$, $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$.

Example 2

Find the exact values of $\sin\frac{5\pi}{3}$, $\cos\frac{5\pi}{3}$ and $\tan\frac{5\pi}{3}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Reference angle	$\frac{\pi}{3}$
Step 2 Signs: fourth quadrant	$\sin < 0, \cos > 0, \tan < 0$
Step 3 Values	$-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}$

APPLY

4 $\cos\frac{7\pi}{6}$

5 $\tan\frac{3\pi}{4}$

6 $\sin\left(-\frac{\pi}{3}\right)$

I.3 Another angle, the same sine or cosine

Example 3

Find α, $0 \leq \alpha \leq 2\pi$, $\alpha \neq \frac{\pi}{5}$, with $\sin\alpha = \sin\frac{\pi}{5}$; then $\beta \neq \frac{\pi}{5}$ with $\cos\beta = \cos\frac{\pi}{5}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Same sine: reflect in the y-axis	$\alpha = \pi - \frac{\pi}{5} = \frac{4\pi}{5}$
Step 2 Same cosine: reflect in the x-axis	$\beta = 2\pi - \frac{\pi}{5} = \frac{9\pi}{5}$

APPLY

7 $\sin\alpha = \sin\frac{2\pi}{7}$, another α in $[0, 2\pi]$.

8 $\cos\beta = \cos\frac{3\pi}{8}$, another β in $[0, 2\pi]$.

I.4 The ambiguous case of the sine rule

Property 2 *the ambiguous case of the sine rule*

Two sides and an angle *not* between them: $\sin C = k$, $0 < k < 1$, gives C and $180° - C$. Both are possible when each leaves room for the third angle.

Watch out 1

The calculator gives only the acute angle for arcsin. Always ask whether $180° - C$ also works.

Example 4

In the triangle ABC, $AB = 9$ cm, $AC = 6$ cm and $A\hat{B}C = 36°$. Find the possible values of $A\hat{C}B$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Sine rule for the angle	$\sin C = \dfrac{9 \sin 36°}{6} = 0.881\ldots$
Step 2 The acute angle	$C = 61.8°$
Step 3 The obtuse angle	$C = 180° - 61.8° = 118.2°$
Step 4 Check the angle sum	$36° + 118.2° < 180°$: both triangles exist

APPLY

9 $a = 5$, $b = 7$, $\hat{A} = 30°$: the possible values of $\hat{B}$.

10 $a = 9$, $b = 6$, $\hat{A} = 50°$: how many triangles?

II IDENTITIES

SL 3.6 · $\cos^2\theta + \sin^2\theta = 1$, double angles, ratios

II.1 $\cos^2\theta + \sin^2\theta = 1$

Property 3 *the Pythagorean identity*

$P(\cos\theta, \sin\theta)$ is on the unit circle: $\cos^2\theta + \sin^2\theta = 1$ for every θ. From one ratio and the quadrant, it gives the others.

Watch out 2

$\sin\theta = \pm\sqrt{1-\cos^2\theta}$: the quadrant chooses the sign. $\cos^2\theta$ means $(\cos\theta)^2$.

Example 5

Given $\sin\theta = -\dfrac{7}{25}$, with $\pi < \theta < \dfrac{3\pi}{2}$, find $\cos\theta$ and $\tan\theta$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 $\cos^2\theta = 1 - \sin^2\theta$	$1 - \dfrac{49}{625} = \dfrac{576}{625}$
Step 2 Square root, sign from the quadrant	third quadrant: $\cos\theta = -\dfrac{24}{25}$
Step 3 $\tan = \dfrac{\sin}{\cos}$	$\tan\theta = \dfrac{7}{24}$

APPLY

1 $\cos\theta = \frac{1}{3}$, $\frac{3\pi}{2} < \theta < 2\pi$: find $\sin\theta$.

2 $\sin\theta = \frac{2}{3}$, θ obtuse: find $\cos\theta$.

II.2 Double angles

Property 4 *the double angle identities*

$\sin 2\theta = 2\sin\theta\cos\theta$; $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$: choose the form that keeps the ratio you have.

Example 6

Given $\cos\theta = \dfrac{12}{13}$, θ acute, find the exact values of $\sin 2\theta$ and $\cos 2\theta$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 $\sin\theta$ first, positive	$\sin\theta = \sqrt{1 - \frac{144}{169}} = \dfrac{5}{13}$
Step 2 $\sin 2\theta = 2\sin\theta\cos\theta$	$2 \times \dfrac{5}{13} \times \dfrac{12}{13} = \dfrac{120}{169}$
Step 3 $\cos 2\theta = 2\cos^2\theta - 1$	$\dfrac{288}{169} - 1 = \dfrac{119}{169}$

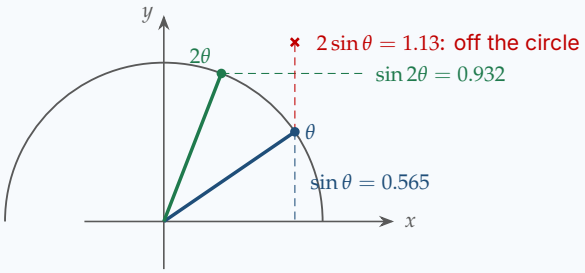


Figure 2 — $\theta = 0.6$: $\sin 2\theta = 0.932 = 2 \times 0.565 \times 0.825$; the point at 2θ is not twice as high.

APPLY

3 $\cos\theta = 0.2$: find $\cos 2\theta$.

4 $\sin\theta = \frac{1}{4}$: find $\cos 2\theta$.

II.3 Proving an identity

Example 7

Show that $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Start from the LHS: difference of squares	$(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$
Step 2 $\cos^2\theta + \sin^2\theta = 1$	$= \cos^2\theta - \sin^2\theta$
Step 3 A form of $\cos 2\theta$	$= \cos 2\theta = \text{RHS}$

APPLY

5 Show that $\dfrac{1 - \cos 2\theta}{2} \equiv \sin^2\theta$.

6 Show that $\dfrac{\sin 2\theta}{\sin\theta} \equiv 2\cos\theta$.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A The unit circle

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 5]

The point P lies on the unit circle, and the angle from the positive x-axis to OP is $\frac{5\pi}{6}$.

(a) Write down the exact coordinates of P. [2]

(b) Find the exact value of $\tan\frac{5\pi}{6}$. [2]

(c) Write down another value of α, $0 \leq \alpha \leq 2\pi$, such that $\sin\alpha = \sin\frac{5\pi}{6}$. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Reference angle and signs: the coordinates	$x = -\frac{\sqrt{3}}{2}$	A1
	$y = \frac{1}{2}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $\tan = \frac{\sin}{\cos}$	$\frac{1/2}{-\sqrt{3}/2}$	M1
Step 2 Simplify	$-\frac{1}{\sqrt{3}}$	A1
	Or $-\frac{\sqrt{3}}{3}$.	

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Same sine: $\pi - \theta$	$\frac{\pi}{6}$	A1

B The ambiguous case (Paper 2)

EXAM-STYLE QUESTION 2 Paper 2 · calculator · [Maximum mark: 6]

In a triangle ABC, $AB = 10$ cm, $AC = 7$ cm and $A\hat{B}C = 40°$.

(a) Find the two possible values of $A\hat{C}B$. [4]

(b) Find the smaller of the two possible areas of the triangle ABC. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Sine rule for the angle	$\dfrac{\sin C}{10} = \dfrac{\sin 40°}{7}$	M1
Step 2 The acute angle	$A\hat{C}B = 66.7°$ $(66.6741\ldots)$	A1
Step 3 The obtuse angle: $180° - C$	$180° - 66.674\ldots$	M1
	The second angle with the same sine.	
	$A\hat{C}B = 113°$ $(113.325\ldots)$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The third angle	$B\hat{A}C = 180° - 40° - 113.325\ldots° = 26.674\ldots°$, $\dfrac{1}{2} \times 10 \times 7 \sin 26.674\ldots°$	(M1)
	The smaller area comes with the obtuse C.	
Step 2 $\frac{1}{2}ab\sin C$	$15.7\ \text{cm}^2$ $(15.7120\ldots)$	A1

C Double angles

EXAM-STYLE QUESTION 3 Paper 1 · no calculator · [Maximum mark: 7]

It is given that $\cos\theta = -\frac{2}{3}$, where $\frac{\pi}{2} < \theta < \pi$.

(a) Find the exact value of $\sin\theta$. [3]

(b) Find the exact value of $\sin 2\theta$. [2]

(c) Find the exact value of $\cos 2\theta$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $\sin^2\theta = 1 - \cos^2\theta$	$\sin^2\theta = 1 - \frac{4}{9} = \frac{5}{9}$	M1
Step 2 Sign from the quadrant	$\sin\theta > 0$ because $\frac{\pi}{2} < \theta < \pi$	R1
Step 3 Square root	$\sin\theta = \frac{\sqrt{5}}{3}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $\sin 2\theta = 2\sin\theta\cos\theta$	$2 \times \frac{\sqrt{5}}{3} \times \left(-\frac{2}{3}\right)$	M1
Step 2 Simplify	$-\frac{4\sqrt{5}}{9}$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $\cos 2\theta = 2\cos^2\theta - 1$	$2\left(-\frac{2}{3}\right)^2 - 1$ *Or* $1 - 2\sin^2\theta$, *or* $\cos^2\theta - \sin^2\theta$.	M1
Step 2 Simplify	$-\frac{1}{9}$	A1

D An identity, then an equation (Paper 2)

EXAM-STYLE QUESTION 4 Paper 2 · calculator · [Maximum mark: 6]

Let $f(x) = (\sin x + \cos x)^2$.

(a) Show that $f(x) = 1 + \sin 2x$. [2]

(b) Write down the maximum value of $f(x)$. [1]

(c) Solve $f(x) = 1.5$ for $0 \leq x \leq \pi$. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Expand the square	$\sin^2 x + 2\sin x \cos x + \cos^2 x$	M1
Step 2 $\cos^2 x + \sin^2 x = 1$	$= 1 + 2\sin x \cos x$	A1
	Both identities must be seen.	
Step 3 $2\sin x \cos x = \sin 2x$	$= 1 + \sin 2x$	AG

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The maximum of $\sin 2x$ is 1	2	A1
	The maximum of $\sin 2x$ *is* 1.	

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Write the equation in $\sin 2x$	$\sin 2x = 0.5$	(M1)
	Or the intersection of the graphs.	
Step 2 Solve for $2x$, then x	$x = 0.262 \ (\frac{\pi}{12})$	A1
	$x = 1.31 \ (\frac{5\pi}{12})$	A1

IV WHICH METHOD? READ THE QUESTION

3.6 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"exact value" of sin, cos, tan *of a multiple of* $\frac{\pi}{6}$ *or* $\frac{\pi}{4}$	the reference angle the sign in the quadrant	a surd or a fraction, no decimal P1
two sides and an angle not between them	the ambiguous case	C and $180° - C$; check the angle sum P2
one ratio given, another wanted	$\cos^2\theta + \sin^2\theta = 1$	the square root, then the sign from the quadrant P3
$\tan\theta$ *wanted*	$\tan\theta = \frac{\sin\theta}{\cos\theta}$	divide the two ratios D1
$\sin 2\theta$	$2\sin\theta\cos\theta$	both ratios needed P4
$\cos 2\theta$	three forms	$2\cos^2\theta - 1$ when only $\cos\theta$ is known P4
"show that" with trigonometry	start from the more complicated side	expand, then use an identity

In the exam 1 *command terms*

Exact value: surds and fractions, no calculator decimals. **Write down**: no working needed. **Show that**: start from one side and end exactly on the other.

V ON THE GDC: PAPER 2

SL 3.5 · 3.6 · NumWorks · every calculator move of this topic

1 Checking an exact value

Calculation → tan(5π/6) → EXE

In radians, compute the ratio and the exact answer you found: the two decimals must agree. A useful check after a Paper 1 question.

CALCULATION

```
tan(5π/6)
-0.5773502692
-1/√3
-0.5773502692
```

2 Both angles of the ambiguous case

Calculation → shift → sin → 10×sin(40)/7 → EXE → 180−Ans

In degrees. arcsin gives the acute angle; 180−Ans gives the obtuse one. **On paper:** both values, and why both are possible.

CALCULATION

```
arcsin(10*sin(40)/7)
66.67417652
180-Ans
113.3258235
```

3 Solving with an identity, on the graph

Grapher → f(x)=(sin(x)+cos(x))^2 → g(x)=1.5 → Find → Intersection

In radians, window $0 \le x \le \pi$. The identity gives the exact answers $\frac{\pi}{12}$ and $\frac{5\pi}{12}$; the graph confirms them.

GRAPHER · INTERSECTION

```
f(x)=(sin(x)+cos(x))^2
g(x)=1.5
x=0.2617994 y=1.5
x=1.308997 y=1.5
```

ANSWERS TO THE APPLY QUESTIONS

I.1 $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

I.2 $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

I.3 $(0, -1)$

I.4 $-\frac{\sqrt{3}}{2}$

I.5 -1

I.6 $-\frac{\sqrt{3}}{2}$

I.7 $\frac{5\pi}{7}$

I.8 $\frac{13\pi}{8}$

I.9 44.4° or 135.6°

I.10 one: $\hat{B} = 30.7^\circ$ (149.3° is too big)

II.1 $-\frac{2\sqrt{2}}{3}$

II.2 $-\frac{\sqrt{5}}{3}$

II.3 -0.92

II.4 $\frac{7}{8}$

II.5 $\dfrac{1-(1-2\sin^2\theta)}{2}$

II.6 $\dfrac{2\sin\theta\cos\theta}{\sin\theta}$

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