Trigonometric equations

An equation such as $\sin x = \frac{1}{2}$ has infinitely many solutions; an exam question always gives an interval and wants all of the solutions in it. One value from the calculator or the unit circle, then the symmetry of the graph for the others, then the period to stay inside the interval.

I SOLVING IN AN INTERVAL

SL 3.8 · $\sin x = k$, $\cos x = k$, $\tan x = k$, and $\sin(bx) = k$

I.1 The principal value

Definition 1 *principal value*

$\arcsin k \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ solves $\sin x = k$; $\arccos k \in [0, \pi]$ solves $\cos x = k$; $\arctan k \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ solves $\tan x = k$. The calculator gives only this one.

Example 1

Find $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$, $\arccos\left(-\frac{1}{2}\right)$ and $\arctan\sqrt{3}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 The exact value, ignoring the sign	$\frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{3}$
Step 2 Place it in the principal interval	$-\frac{\pi}{4}, \pi - \frac{\pi}{3} = \frac{2\pi}{3}, \frac{\pi}{3}$

APPLY

1. $\arcsin\frac{1}{2}$
2. $\arccos 0$
3. $\arctan(-1)$

I.2 $\sin x = k$ in an interval

Property 1 *all the solutions*

With α the principal value: $\sin x = k$: $x = \alpha$ or $\pi - \alpha$; $\cos x = k$: $x = \pm\alpha$; $\tan x = k$: $x = \alpha$; each $+2k\pi$ (tan: $+k\pi$). Keep those in the interval.

Example 2

Solve $\sin x = -\dfrac{\sqrt{3}}{2}$, for $0 \le x \le 2\pi$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Principal value	$\alpha = -\dfrac{\pi}{3}$, not in the interval
Step 2 The other one: $\pi - \alpha$	$\pi + \dfrac{\pi}{3} = \dfrac{4\pi}{3}$
Step 3 Add 2π to α	$-\dfrac{\pi}{3} + 2\pi = \dfrac{5\pi}{3}$
Step 4 List the solutions	$x = \dfrac{4\pi}{3}$ or $x = \dfrac{5\pi}{3}$

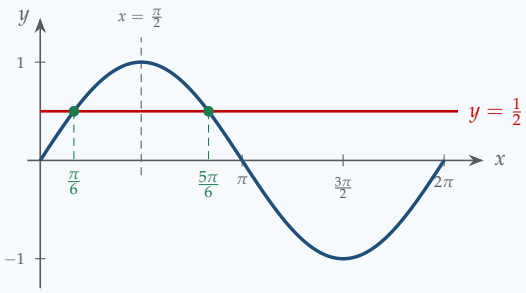


Figure 1 — $\sin x = \frac{1}{2}$ on $[0, 2\pi]$: two solutions, $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, symmetric about $x = \frac{\pi}{2}$; Example 2's level $-\frac{\sqrt{3}}{2}$ gives two, symmetric about $\frac{3\pi}{2}$.

APPLY

4 Solve $\sin x = \frac{\sqrt{2}}{2}$, $0 \le x \le 2\pi$.

5 Solve $\sin x = 0.4$, $0 \le x \le 2\pi$.

I.3 $\cos x = k$

Example 3

Solve $\cos x = -\dfrac{\sqrt{2}}{2}$, for $0 \le x \le 2\pi$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Principal value	$\alpha = \dfrac{3\pi}{4}$
Step 2 The other one: $-\alpha + 2\pi$	$2\pi - \dfrac{3\pi}{4} = \dfrac{5\pi}{4}$
Step 3 List the solutions	$x = \dfrac{3\pi}{4}$ or $x = \dfrac{5\pi}{4}$

APPLY

6 Solve $\cos x = \frac{1}{2}, 0 \leq x \leq 2\pi$.

7 Solve $\cos x = -0.6, 0 \leq x \leq 2\pi$.

I.4 $\tan x = k$

Example 4

Solve $\tan x = \dfrac{1}{\sqrt{3}}$, for $0 \leq x \leq 2\pi$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Principal value	$\alpha = \dfrac{\pi}{6}$
Step 2 Period π: add π	$\dfrac{\pi}{6} + \pi = \dfrac{7\pi}{6}$
Step 3 List the solutions	$x = \dfrac{\pi}{6}$ or $x = \dfrac{7\pi}{6}$

APPLY

8 Solve $\tan x = -\frac{1}{\sqrt{3}}, 0 \leq x \leq 2\pi$.

9 Solve $\tan x = 2, 0 \leq x \leq 2\pi$.

I.5 A multiple angle

Property 2 *a multiple angle*

$\sin(bx) = k, 0 \leq x \leq 2\pi$: let $u = bx$, solve $\sin u = k$ for $0 \leq u \leq 2b\pi$, then divide by b: b times as many solutions.

Watch out 1

Change the interval with the variable: for $\sin 2x$ on $[0, 2\pi]$, $2x$ runs over $[0, 4\pi]$.

Example 5

Solve $\sin 3x = 1$, for $0 \leq x \leq 2\pi$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 New interval for $3x$	$0 \leq 3x \leq 6\pi$
Step 2 Solve for $3x$	$3x = \dfrac{\pi}{2}, \dfrac{5\pi}{2}, \dfrac{9\pi}{2}$
Step 3 Divide by 3	$x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}$

APPLY

10 Solve $\cos 2x = -1, 0 \leq x \leq 2\pi$.

11 Solve $\tan 2x = 1, 0 \leq x \leq \pi$.

II QUADRATIC EQUATIONS IN SIN, COS, TAN

SL 3.8 · substitution, identities, rejected roots

II.1 A quadratic in one ratio

Property 3 *one ratio, one quadratic*

Use an identity so that one ratio remains ($\cos^2 x = 1 - \sin^2 x$, $\cos 2x = 1 - 2\sin^2 x$), set $s = \sin x$: a quadratic in s. Solve it, then $\sin x = s$ for each root.

Watch out 2

Never divide by $\sin x$: factorise, or the solutions of $\sin x = 0$ are lost. A root with $|s| > 1$ gives no solution: say so.

Example 6

Solve $2\cos^2 x + \cos x - 1 = 0$, for $0 \leq x \leq 2\pi$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Factorise in $\cos x$	$(2\cos x - 1)(\cos x + 1) = 0$
Step 2 Two levels	$\cos x = \dfrac{1}{2}$ or $\cos x = -1$
Step 3 Solve each	$x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$; $x = \pi$

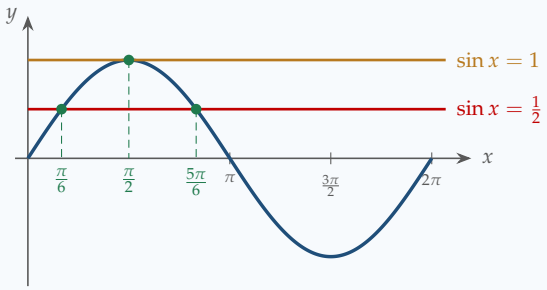


Figure 2 — Exam-style question 3: $2\sin^2 x - 3\sin x + 1 = 0$, levels $\frac{1}{2}$ (two solutions) and 1 (one, where the curve touches).

APPLY

1 Solve $\tan^2 x = 1$, $0 \leq x \leq \pi$.

2 Solve $2\sin^2 x + \sin x = 0$, $0 \leq x \leq 2\pi$.

II.2 An identity first

Example 7

Solve $2\sin^2 x = 3\cos x$, for $0 \leq x \leq 2\pi$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 One ratio: $\sin^2 x = 1 - \cos^2 x$	$2 - 2\cos^2 x = 3\cos x$
Step 2 Rearrange, factorise	$(2\cos x - 1)(\cos x + 2) = 0$
Step 3 Reject the impossible level	$\cos x = -2$: impossible
Step 4 Solve the other	$\cos x = \frac{1}{2}$: $x = \frac{\pi}{3}, \frac{5\pi}{3}$

APPLY

3 Solve $\cos 2x + \sin x = 0, 0 \leq x \leq 2\pi$.

4 Solve $3\sin 2x = 2\sin x, 0 \leq x \leq 2\pi$.

II.3 An equation for the GDC

Example 8

Solve $2\sin x = x - 1$, for $0 \leq x \leq 2\pi$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Graph both sides	$y = 2\sin x$ and $y = x - 1$
Step 2 Intersections in the interval	one: $x = 2.38006\ldots$
Step 3 Answer, 3 s.f.	$x = 2.38$

APPLY

5 Solve $3\cos x = x, 0 \leq x \leq 2\pi$.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A Basic equations

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 5]

This question is about solving equations for $0 \leq x \leq 2\pi$.

(a) Solve $2\cos x = \sqrt{3}$. [3]

(b) Solve $\tan x = -1$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Isolate $\cos x$	$\cos x = \dfrac{\sqrt{3}}{2}$	(M1)
Step 2 Principal value	$x = \dfrac{\pi}{6}$	A1
Step 3 The other one: $2\pi - \alpha$	$x = \dfrac{11\pi}{6}$	A1

$2\pi - \frac{\pi}{6}$.

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Principal value, placed in the interval	$x = \dfrac{3\pi}{4}$	A1
Step 2 Add π	$x = \dfrac{7\pi}{4}$	A1

Add the period π.

B A multiple angle

EXAM-STYLE QUESTION 2 — Paper 1 · no calculator · [Maximum mark: 4]

Solve $2\sin(2x) = 1$, for $0 \leq x \leq 2\pi$.

WORKING

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Isolate $\sin 2x$, new interval for $2x$	$\sin 2x = \frac{1}{2},\ 0 \leq 2x \leq 4\pi$	M1
	The interval for $2x$.	
Step 2 Solve for $2x$	$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$	A1
Step 3 Divide by 2	$x = \frac{\pi}{12}, \frac{5\pi}{12}$	A1
	$x = \frac{13\pi}{12}, \frac{17\pi}{12}$	A1

C A quadratic in one ratio

EXAM-STYLE QUESTION 3 — Paper 1 · no calculator · [Maximum mark: 7]

Consider the equation $2\cos^2 x + 3\sin x - 3 = 0$, for $0 \leq x \leq 2\pi$.

(a) Show that the equation can be written as $2\sin^2 x - 3\sin x + 1 = 0$. [2]

(b) Hence solve the equation. [5]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 One ratio: $\cos^2 x = 1 - \sin^2 x$	$2(1 - \sin^2 x) + 3\sin x - 3 = 0$	M1
Step 2 Simplify	$-2\sin^2 x + 3\sin x - 1 = 0$	A1
Step 3 Multiply by -1	$2\sin^2 x - 3\sin x + 1 = 0$	AG

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Factorise	$(2\sin x - 1)(\sin x - 1) = 0$	M1
Step 2 First level	$\sin x = \frac{1}{2}$	A1
Step 3 Second level	$\sin x = 1$	A1
Step 4 Solve $\sin x = \frac{1}{2}$	$x = \frac{\pi}{6}, \frac{5\pi}{6}$	A1
Step 5 Solve $\sin x = 1$	$x = \frac{\pi}{2}$	A1

D With the GDC (Paper 2)

EXAM-STYLE QUESTION 4 Paper 2 · calculator · [Maximum mark: 6]

Consider the equation $3\sin x + \cos 2x = 1.5$.

(a) Solve the equation for $0 \leq x \leq 2\pi$. [3]

(b) Show that the equation can be written as $2\sin^2 x - 3\sin x + \frac{1}{2} = 0$. Hence explain why it has exactly two solutions for $0 \leq x \leq 2\pi$. [2]

(c) Write down the number of solutions of the equation for $0 \leq x \leq 4\pi$. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Graph both sides	the graphs of $y = 3\sin x + \cos 2x$ and $y = 1.5$	(M1)
Step 2 Intersections	$x = 0.192$ $(0.192163\ldots)$	A1
	$x = 2.95$ $(2.94943\ldots)$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $\cos 2x = 1 - 2\sin^2 x$, rearrange	$3\sin x + 1 - 2\sin^2 x = 1.5$, so $2\sin^2 x - 3\sin x + \frac{1}{2} = 0$	M1
Step 2 Reject the impossible level	$\sin x = 0.191$ or $\sin x = 1.31$; $\sin x = 1.31$ is impossible, and $\sin x = 0.191$ gives two solutions	R1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Count the intersections on $[0, 4\pi]$	4	A1

Two solutions in each period of 2π.

IV WHICH METHOD? READ THE QUESTION

3.8 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
$\sin x = k$ *in an interval*	α and $\pi - \alpha$	then $\pm 2\pi$ until the interval is covered P1
$\cos x = k$	α and $-\alpha$ (or $2\pi - \alpha$)	symmetric about $x = 0$ and $x = \pi$ P1
$\tan x = k$	period π	α, $\alpha + \pi$, ... P1
$\sin(bx) = k$, $0 \le x \le 2\pi$	$0 \le bx \le 2b\pi$	solve for bx, then divide by b P2
$\sin^2 x$ *and* $\cos x$ *together*	$\cos^2 x + \sin^2 x = 1$	one ratio only, then a quadratic P3
$\cos 2x$ *and* $\sin x$ *together*	$\cos 2x = 1 - 2\sin^2 x$	a quadratic in $\sin x$ P3
a root such as $\sin x = 2$	$-1 \le \sin x \le 1$	reject it, and say why

In the exam 1 *command terms*

Solve: all the solutions in the interval, exact in Paper 1. **Hence**: use the result of the previous part. **Show that**: every step to the given form.

V ON THE GDC: PAPER 2

SL 3.8 · NumWorks · every calculator move of this topic

1 Radians, and the whole interval

Settings → Radians → Grapher → Axes → x: 0 to 2π

The window must cover the whole interval of the question: a solution outside the window is a lost mark.

GRAPHER · AXES

```
Angle measure: Radians
X: 0 .. 6.283 Y: -3 .. 3
```

2 Every intersection, one by one

Grapher → f(x)=3sin(x)+cos(2x) → g(x)=1.5 → Find → Intersection

Move along the curve to the next intersection until the end of the window. **On paper:** the equation, then every solution to 3 s.f.

GRAPHER · INTERSECTION

```
x=0.1921635 y=1.5
x=2.949429 y=1.5
```

3 A quadratic in $\sin x$

Equations → 2x^2−3x+0.5=0 → Solve

Here x stands for $\sin x$. A root above 1 is rejected; the other gives arcsin and $\pi -$ arcsin.

EQUATIONS

```
2x^2−3x+0.5=0
x1 = 0.190983
x2 = 1.309017
```

ANSWERS TO THE APPLY QUESTIONS

I.1 $\frac{\pi}{6}$

I.2 $\frac{\pi}{2}$

I.3 $-\frac{\pi}{4}$

I.4 $\frac{\pi}{4}, \frac{3\pi}{4}$

I.5 $0.412, 2.73$

I.6 $\frac{\pi}{3}, \frac{5\pi}{3}$

I.7 $2.21, 4.07$

I.8 $\frac{5\pi}{6}, \frac{11\pi}{6}$

I.9 $1.11, 4.25$

I.10 $\frac{\pi}{2}, \frac{3\pi}{2}$

I.11 $\frac{\pi}{8}, \frac{5\pi}{8}$

II.1 $\frac{\pi}{4}, \frac{3\pi}{4}$

II.2 $0,\ \pi,\ 2\pi,\ \frac{7\pi}{6},\ \frac{11\pi}{6}$

II.3 $\frac{\pi}{2},\ \frac{7\pi}{6},\ \frac{11\pi}{6}$

II.4 $0,\ \pi,\ 2\pi;\ 1.23, 5.05$

II.5 $x = 1.17$

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