Areas between curves

An area is never negative, but an integral can be. Below the x-axis the integral counts the area with a minus sign, and between two curves it counts the gap, top minus bottom. Every area question starts the same way: find where the curves cross, and which one is on top.

I THE AREA BETWEEN A CURVE AND THE X-AXIS

SL 5.5 · 5.11 · regions above and below the axis

I.1 A region above the axis

Property 1 *above the axis*

$f(x) \geq 0$ on $[a, b]$: the area between the graph, the x-axis and $x = a$, $x = b$ is $A = \int_a^b f(x)\,\mathrm{d}x$.

Example 1

Find the area of the region enclosed by $y = 4x - x^2$ and the x-axis.

WHAT YOU DO	WHAT YOU WRITE
Step 1 The roots are the limits	$x(4-x) = 0$: $x = 0, x = 4$
Step 2 Integrate between them	$\left[2x^2 - \frac{x^3}{3}\right]_0^4$
Step 3 Compute	$32 - \frac{64}{3} = \frac{32}{3}$

APPLY

1 Under $y = 2x + 1$, from $x = 1$ to $x = 3$.

2 Enclosed by $y = 1 - x^2$ and the x-axis.

I.2 A region below the axis

Property 2 *below the axis*

$f(x) \leq 0$ on $[a, b]$: the integral is negative, the area is its size, $\left|\int_a^b f\right|$. When the graph crosses the axis, split at each root and add the sizes: $\int_a^b |f(x)|\,\mathrm{d}x$.

Example 2

Find the area of the region enclosed by $y = x^2 - 9$ and the x-axis.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Roots	$x = \pm 3$
Step 2 Integrate	$\left[\frac{x^3}{3} - 9x\right]_{-3}^{3} = -36$
Step 3 Below the axis: the size	area 36

APPLY

3 Enclosed by $y = x^2 - 3x$ and the x-axis.

4 Between $y = -e^x$, the axis, $x = 0$ and $x = 1$.

I.3 Above and below: split at the roots

Watch out 1

$\int_0^3 (x^2 - 4x + 3)\,dx = 0$ does not mean no area: the parts cancel. Roots first, then each piece.

Example 3

Find the total area between $y = x^2 - x - 2$, the x-axis, $x = 0$ and $x = 3$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Roots in the interval	$(x-2)(x+1) = 0$: $x = 2$
Step 2 Each piece	$\int_0^2 = -\frac{10}{3}$, $\int_2^3 = \frac{11}{6}$
Step 3 Add the sizes	$\frac{10}{3} + \frac{11}{6} = \frac{31}{6}$

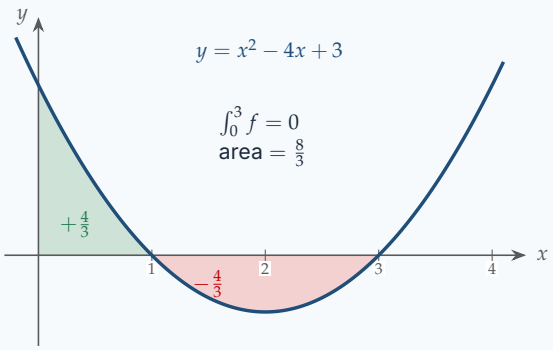


Figure 1 — $f(x) = x^2 - 4x + 3$ from 0 to 3: $\frac{4}{3}$ above, $\frac{4}{3}$ below; the integral is 0, the area $\frac{8}{3}$.

APPLY

5 Total area between $y = x^3 - 9x$ and the axis, $-3 \le x \le 3$.

6 Total area between $y = \sin x$ and the axis, $0 \le x \le 2\pi$.

II.1 Top minus bottom

Property 3 *between two curves*

$f(x) \geq g(x)$ on $[a, b]$: $A = \int_a^b \big(f(x) - g(x)\big)\,\mathrm{d}x$, top minus bottom, above or below the axis; a and b usually solve $f(x) = g(x)$.

Example 4

Find the area enclosed by $y = x^2$ and $y = x + 6$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Intersections	$x^2 - x - 6 = 0$: $x = -2, x = 3$
Step 2 Top minus bottom	$\int_{-2}^{3} (x + 6 - x^2)\,\mathrm{d}x$
Step 3 Antiderivative	$\left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3}$
Step 4 Compute	$\frac{27}{2} + \frac{22}{3} = \frac{125}{6}$

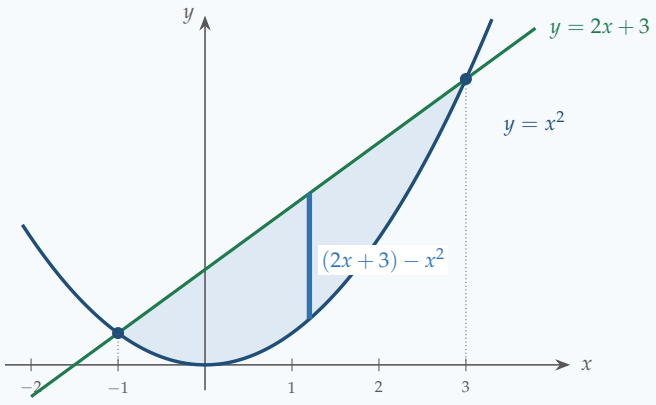


Figure 2 — $y = 2x + 3$ and $y = x^2$: thin strips of height top minus bottom make $\frac{32}{3}$.

APPLY

1 Enclosed by $y = x^2$ and $y = 4x$.

2 Enclosed by $y = \sqrt{x}$ and $y = x$.

II.2 When the top curve changes

Watch out 2

When the curves cross inside the interval, split at the intersection and take top minus bottom on each piece.

Example 5

Find the total area between $y = x^3$ and $y = 4x$ for $-2 \le x \le 2$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Where they cross	$x = -2, 0, 2$
Step 2 Top on each piece	$x^3 \ge 4x$ on $[-2, 0]$; $4x \ge x^3$ on $[0, 2]$
Step 3 Integrate each, add	$4 + 4 = 8$

APPLY

3 Between $y = \sin x$ and $y = \cos x$ for $0 \le x \le \frac{\pi}{2}$.

II.3 An unknown: find k

Example 6

The area under $y = x^2$ from $x = 0$ to $x = a$ is 9. Find a.

WHAT YOU DO	WHAT YOU WRITE
Step 1 The area in terms of a	$\left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$
Step 2 Write the equation	$\frac{a^3}{3} = 9$
Step 3 Solve	$a = 3$

APPLY

4 The area under $y = k\sqrt{x}$ from 0 to 4 is 32. Find k.

III.1 Intersections and $\int |f-g|$ on the GDC

Property 4 *the GDC method*

Intersections on the GDC (full values stored), then $A = \int_a^b |f(x) - g(x)|\,\mathrm{d}x$: no need to know which curve is on top.

Watch out 3

Use the stored intersections, not their rounded values, and write the integral you typed, with its limits.

Example 7

Find the area enclosed by $y = 3 - x^2$ and $y = 2^x$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Intersections on the GDC	$x = -1.64$ $(-1.636576\ldots)$ and $x = 1$
Step 2 Write the integral	$\int_{-1.636\ldots}^{1} \left\|3 - x^2 - 2^x\right\| \mathrm{d}x$
Step 3 Compute	3.69

APPLY

1 The area between $y = x\,\mathrm{e}^{-x}$, the x-axis and $x = 3$.

IV EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A A curve and the axis

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 5]

Let $f(x) = 6x - 3x^2$. The region R is enclosed by the graph of f and the x-axis.

(a) Find the x-intercepts of the graph of f. [2]

(b) Find the area of R. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Factorise: the roots	$3x(2 - x) = 0$	(M1)
Step 2 Write them	$x = 0$ and $x = 2$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Integrate between the roots	$\int_0^2 (6x - 3x^2)\,\mathrm{d}x$ *The limits are the intercepts.*	M1
Step 2 Antiderivative, top minus bottom	$\left[3x^2 - x^3\right]_0^2 = 12 - 8$	A1
Step 3 Compute	$= 4$	A1

EXAM-STYLE QUESTION 2

Paper 1 · no calculator · [Maximum mark: 6]

Let $f(x) = x^2 - 4x + 3$.

(a) Find the x-intercepts of the graph of f. [2]

(b) Find the total area of the regions enclosed by the graph of f, the x-axis and the lines $x = 0$ and $x = 3$. [4]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Factorise: the roots	$(x-1)(x-3) = 0$	(M1)
Step 2 Write them	$x = 1$ and $x = 3$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Piece above the axis	$\int_0^1 f(x)\,\mathrm{d}x = \left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^1 = \frac{4}{3}$ *Above the axis.*	A1
Step 2 Piece below the axis	$\int_1^3 f(x)\,\mathrm{d}x = 0 - \frac{4}{3} = -\frac{4}{3}$ *Below the axis: negative.*	A1
Step 3 Add the sizes	$\text{area} = \frac{4}{3} + \left\lvert -\frac{4}{3} \right\rvert$ *Add the sizes.*	M1
Step 4 Compute	$= \frac{8}{3}$ *$\int_0^3 f = 0$ earns no marks.*	A1

B Two curves

EXAM-STYLE QUESTION 3

Paper 1 · no calculator · [Maximum mark: 6]

The graphs of $y = x^2$ and $y = 2x + 3$ intersect at the points A and B.

(a) Find the x-coordinates of A and B. [3]

(b) Find the area of the region enclosed by the two graphs. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Intersections: $f = g$	$x^2 = 2x + 3$	M1
Step 2 Factorise	$(x - 3)(x + 1) = 0$	(A1)
Step 3 Write them	$x = -1,\ x = 3$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Top minus bottom	$\int_{-1}^{3} (2x + 3 - x^2)\,\mathrm{d}x$	M1
	Top minus bottom: the line is above between the intersections.	
Step 2 Antiderivative, top minus bottom	$\left[x^2 + 3x - \frac{x^3}{3}\right]_{-1}^{3} = 9 - \left(-\frac{5}{3}\right)$	A1
Step 3 Compute	$= \frac{32}{3}$	A1

EXAM-STYLE QUESTION 4

Paper 1 · no calculator · [Maximum mark: 6]

The region R is enclosed by the graphs of $y = \sin x$ and $y = \cos x$ and the lines $x = 0$ and $x = \pi$.

(a) Find the x-coordinate of the point of intersection of the two graphs, for $0 \leq x \leq \pi$. [2]

(b) Find the exact area of R. [4]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $\sin x = \cos x$: $\tan x = 1$	$\sin x = \cos x \Rightarrow \tan x = 1$	(M1)
Step 2 Solve	$x = \frac{\pi}{4}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Split: top minus bottom on each piece	$\int_{0}^{\frac{\pi}{4}} (\cos x - \sin x)\,\mathrm{d}x + \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x)\,\mathrm{d}x$	M1
	Two pieces: cos is on top first, then sin.	
Step 2 First piece	$\left[\sin x + \cos x\right]_{0}^{\frac{\pi}{4}} = \sqrt{2} - 1$	A1
Step 3 Second piece	$\left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\pi} = 1 + \sqrt{2}$	A1
Step 4 Add	$= 2\sqrt{2}$	A1
	$\int_0^{\pi} (\sin x - \cos x)\,\mathrm{d}x = 2$ is not the area.	

C With the GDC (Paper 2)

EXAM-STYLE QUESTION 5 Paper 2 · calculator · [Maximum mark: 6]

Let $f(x) = 4 - x^2$ and $g(x) = \mathrm{e}^x$. The graphs of f and g intersect at two points.

(a) Find the x-coordinates of the points of intersection. [2]

(b) Write down an integral for the area of the region enclosed by the graphs of f and g. [2]

(c) Find the area of this region. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Intersections on the GDC	$x = -1.96 \ \ (-1.96463\ldots)$	A1
	$x = 1.06 \ \ (1.05800\ldots)$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Write the integral	$\int_{-1.96}^{1.06} \left(4 - x^2 - \mathrm{e}^x\right) \mathrm{d}x$	A1A1

A1 for the limits, A1 for top minus bottom (or $\lvert f - g \rvert$).

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Compute	$6.43 \ \ (6.42768\ldots)$	A2

Rounded limits give 6.42 or 6.44: A1.

EXAM-STYLE QUESTION 6 Paper 2 · calculator · [Maximum mark: 6]

Let $f(x) = 2\sin x - \frac{x}{2}$, for $0 \le x \le 2\pi$.

(a) Find the non-zero x-intercept of the graph of f. [2]

(b) Find $\int_0^{2\pi} f(x)\,\mathrm{d}x$. [1]

(c) Find the total area of the regions enclosed by the graph of f and the x-axis, for $0 \le x \le 2\pi$. [2]

(d) Explain why the answers to (b) and (c) are different. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Zero on the GDC	$2\sin x - \frac{x}{2} = 0$ on the GDC	(M1)
Step 2 Read it	$x = 2.47$ $(2.47457\ldots)$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Integral on the GDC	-9.87 $(-9.86960\ldots)$	A1
	Exactly $-\pi^2$.	

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Total area: $\int \lvert f \rvert$	$\int_0^{2\pi} \left\lvert 2\sin x - \frac{x}{2} \right\rvert \mathrm{d}x$	(M1)
Step 2 Compute	$= 14.0$ $(13.9505\ldots)$	A1

WORKING (d)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Signed integral against area	the region below the axis counts negatively in the integral	R1
	A reason about the sign, not only "they are different".	

V WHICH METHOD? READ THE QUESTION

5.11 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"area under the curve", $f \geq 0$	one region, above	$\int_a^b f(x)\,\mathrm{d}x$ P1
"area enclosed by the curve and the x-axis"	the roots	the roots are the limits; check the sign of f between them P1
a region below the axis, or above and below	negative integrals	split at the roots; add the sizes $\left\lvert\int\right\rvert$ P2
"area between the curves"	$f(x) = g(x)$	intersections first; $\int(\text{top} - \text{bottom})$ P3
the curves cross inside the interval	the top curve changes	split at each intersection, top minus bottom on each piece P3
"find k" so that the area is a value	an equation in k	the area in terms of k, then solve
Paper 2, e, ln *or* sin *curves*	the GDC	intersections stored; $\int \lvert f - g\rvert$; the integral written in full P4

In the exam 1 *command terms*

Area: always positive. **Enclosed by**: the curves and lines named, nothing else. **Exact area**: no decimals on Paper 1. **Write down the integral**: the integral with its limits, not its value. **Region** R: shade it on a sketch to see the top and the bottom.

VI ON THE GDC: PAPER 2

SL 5.5 · 5.11 · NumWorks · every calculator move of this topic

1 Intersections first

Grapher → f(x), g(x) → Find → Intersection

Both curves, then each intersection; store the x-values. **On paper:** "$4 - x^2 = e^x$", then $x = -1.96$ and $x = 1.06$.

GRAPHER · INTERSECTION

```
f(x)=4-x^2 g(x)=e^x
x=-1.964636 y=0.140207
x=1.058006 y=2.880622
```

2 The area between the curves

Calculation → Toolbox → Integral → abs(f−g)

Integrate $|f - g|$ between the stored values. $\int_{-1.96}^{1.06} \left(4 - x^2 - \mathrm{e}^x\right) \mathrm{d}x = 6.43$.

On paper:

CALCULATION

```
int(abs(4-x^2-e^x),x,A,B)
6.427687376
```

ANSWERS TO THE APPLY QUESTIONS

I.1 10

I.2 $\frac{4}{3}$

I.3 $\frac{9}{2}$

I.4 $\mathrm{e} - 1$

I.5 $\frac{81}{2}$

I.6 4

II.1 $\frac{32}{3}$

II.2 $\frac{1}{6}$

II.3 $2\sqrt{2} - 2$

II.4 $k = 6$

III.1 0.801

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