Kinematics with integrals

Differentiation goes from s to v to a; integration comes back, a to v to s, and each step costs a constant that the initial conditions give back. The area under the graph of v is a change of position: counted with its sign it is the displacement, counted always positive it is the distance travelled.

I FROM ACCELERATION BACK TO DISPLACEMENT

SL 5.9 · $v = \int a\,\mathrm{d}t$, $s = \int v\,\mathrm{d}t$

I.1 v from a

Property 1 *integrate, then use the initial values*

$$v(t) = \int a(t)\,\mathrm{d}t, \qquad s(t) = \int v(t)\,\mathrm{d}t,$$

each with its constant. The initial velocity $v(0)$ fixes the first constant, the initial displacement $s(0)$ the second. "Starts from rest" means $v(0) = 0$; "starts at O" means $s(0) = 0$.

Example 1

A particle has acceleration $a(t) = 2t + 3\ \mathrm{m\,s^{-2}}$ and initial velocity $-4\ \mathrm{m\,s^{-1}}$. Find $v(t)$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Integrate a, with a constant	$v(t) = t^2 + 3t + c$
Step 2 $v(0)$ fixes c	$v(0) = c = -4$
Step 3 Write $v(t)$	$v(t) = t^2 + 3t - 4$

APPLY

1 $a = 8 - 2t$, from rest: $v(t)$.

2 $a = 6\mathrm{e}^{2t}$, $v(0) = 1$: $v(t)$.

I.2 s from v

Watch out 1

Two constants, two conditions: never assume $c = 0$. "At rest initially" is about v; "at the origin initially" is about s. Read which one the question gives.

Example 2

A particle has velocity $v(t) = 4t^3 - 6t$ m s^{-1}; when $t = 0$ it is 5 m from O. Find $s(t)$, then its displacement when $t = 2$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Integrate v, with a constant	$s(t) = t^4 - 3t^2 + d$
Step 2 $s(0)$ fixes d	$s(0) = d = 5$: $s(t) = t^4 - 3t^2 + 5$
Step 3 Substitute $t = 2$	$s(2) = 16 - 12 + 5 = 9$ m

APPLY

3 $v = 10 - 4t$, starting at O: $s(t)$.

4 $v = 3\cos t$, $s(0) = -2$: $s(t)$.

I.3 From a to v to s

Example 3

$a(t) = 6t + 2$, with $v(0) = -1$ and $s(0) = 3$. Find $s(t)$ and the displacement at $t = 1$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Integrate a	$v(t) = 3t^2 + 2t + c$
Step 2 $v(0)$ fixes c	$c = -1$: $v(t) = 3t^2 + 2t - 1$
Step 3 Integrate v	$s(t) = t^3 + t^2 - t + d$
Step 4 $s(0)$ fixes d	$d = 3$: $s(t) = t^3 + t^2 - t + 3$
Step 5 Substitute $t = 1$	$s(1) = 4$ m

APPLY

5 $a = -4$, $v(0) = 8$, $s(0) = 0$: when is it at rest, and where?

6 $a = 12t$, from rest at O: $s(2)$.

II DISPLACEMENT AND DISTANCE TRAVELLED

SL 5.9 · 5.11 · $\int v\,dt$ and $\int |v|\,dt$

II.1 Displacement as an integral

Property 2 *displacement as an area*

Between two times, the change of displacement is the integral of the velocity:

$$s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t)\,dt.$$

Above the t-axis the particle moves forward, below it moves back.

Example 4

$v(t) = 6 - 2t$ m s^{-1}. Find the displacement from $t = 1$ to $t = 4$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 The integral, with its limits	$\int_1^4 (6 - 2t)\,dt$
Step 2 Antiderivative	$\left[6t - t^2\right]_1^4$
Step 3 Substitute the limits	$8 - 5 = 3$ m

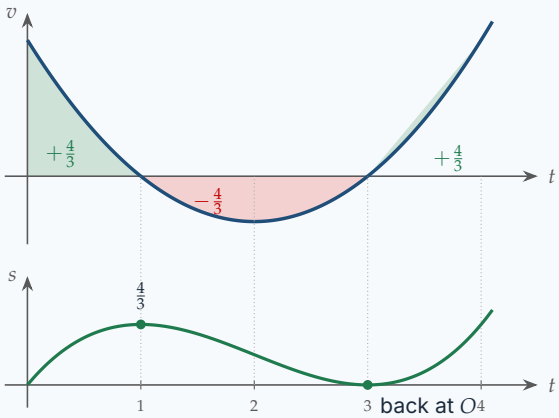


Figure 1 — Exam-style question 3: $v(t) = t^2 - 4t + 3$ and $s(t) = \frac{t^3}{3} - 2t^2 + 3t$, with $s(0) = 0$. The green area adds $\frac{4}{3}$ to s from $t = 0$ to $t = 1$; the red area takes $\frac{4}{3}$ away from $t = 1$ to $t = 3$: at $t = 3$ the particle is back at O.

APPLY

1 $v = 3t^2 + 1$: displacement from $t = 0$ to $t = 2$.

2 $v = 2t - 5$: displacement from $t = 0$ to $t = 5$.

II.2 Distance travelled

Property 3 *distance travelled*

The **distance travelled** from t_1 to t_2 is

$$\int_{t_1}^{t_2} |v(t)|\,\mathrm{d}t.$$

On Paper 1, find the times where $v = 0$, integrate v on each piece and add the sizes. The **displacement** over the same time is $\int_{t_1}^{t_2} v(t)\,\mathrm{d}t$, with its sign.

Example 5

$v(t) = 8 - 2t$ m s^{-1}, for $0 \le t \le 6$. Find the distance travelled and the displacement.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Zeros of v in the interval	$8 - 2t = 0$: $t = 4$
Step 2 Integrate on each piece	$\int_0^4 v\,\mathrm{d}t = 16$, $\int_4^6 v\,\mathrm{d}t = -4$
Step 3 Distance: add the sizes	$16 + 4 = 20$ m
The displacement keeps the signs: $16 - 4 = 12$ m.	

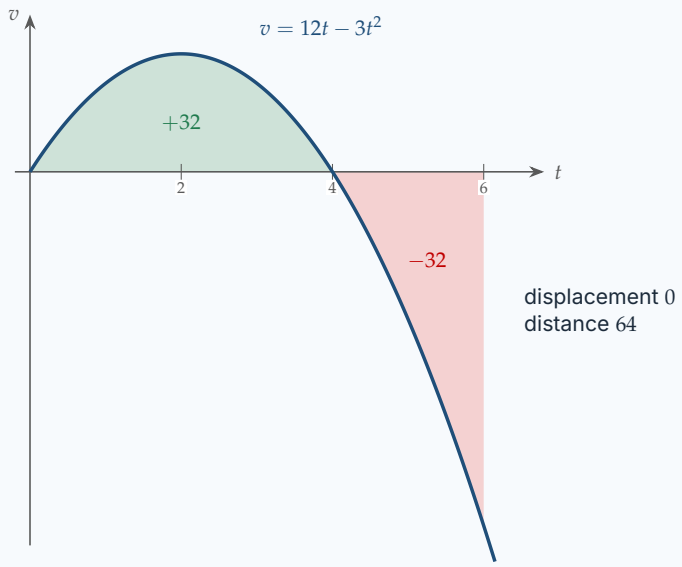


Figure 2 — Exam-style question 4: $v(t) = 12t - 3t^2$, starting at O: forward 32 m from $t = 0$ to $t = 4$, back 32 m from $t = 4$ to $t = 6$. Displacement after 6 s: 0. Distance: 64 m.

Watch out 2

The displacement can be small, even 0, after a long journey. A question on "distance" wants $\int |v|$; on Paper 1 that means splitting at each zero of v inside the interval.

APPLY

3 $v = t - 3$, $0 \le t \le 5$: distance and displacement.

4 $v = 3t^2 - 3$, $0 \le t \le 2$: distance and displacement.

III KINEMATICS ON PAPER 2

SL 5.9 · zeros of v, $\int v$ and $\int |v|$ on the GDC

III.1 Rest, displacement, distance on the GDC

Property 4 *the GDC method*

Paper 2: the zeros of v give the times at rest; $\int_{t_1}^{t_2} v\,\mathrm{d}t$ the displacement; $\int_{t_1}^{t_2} |v|\,\mathrm{d}t$ the distance, with no splitting. To find when the particle is back at its start, solve $\int_0^T v(t)\,\mathrm{d}t = 0$, or $s(T) = s(0)$.

Example 6

$v(t) = 5\sin t - t$ m s^{-1}, for $0 \leq t \leq 4$. Find when the particle is at rest for $t > 0$, its displacement and the distance travelled.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Zeros of v on the GDC $t = 0$ is a zero too: the question asks for $t > 0$.	$t = 2.60$ s $\ (2.59573\ldots)$
Step 2 Displacement: $\int v$	$\displaystyle\int_0^4 (5\sin t - t)\,\mathrm{d}t = 0.268$ m
Step 3 Distance: $\int \lvert v\rvert$	$\displaystyle\int_0^4 \lvert 5\sin t - t\rvert\,\mathrm{d}t = 11.5$ m

APPLY

1 $v = t\cos t$, $0 \leq t \leq 2.5$: time at rest $(t > 0)$, displacement, distance.

III.2 Back to the start

Example 7

A particle starts at O with velocity $v(t) = 4 - \mathrm{e}^{0.5t}$ m s^{-1}. Find the time $T > 0$ at which it is back at O.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Integrate v, $s(0) = 0$	$s(t) = 4t - 2\mathrm{e}^{0.5t} + 2$
Step 2 Back at O: $s(T) = 0$ Or $\int_0^T v\,\mathrm{d}t = 0$, solved on the GDC.	$4T - 2\mathrm{e}^{0.5T} + 2 = 0$
Step 3 Solve on the GDC	$T = 4.67$ s $\ (4.67332\ldots)$

Watch out 3

Integrate v, not s, to get a displacement; and $|v|$, not v, for a distance. Write the integral with its limits before its value: that line carries the method mark.

APPLY

2 $v = 10\mathrm{e}^{-t} - 2$, starting at O: when is it back at O?

IV EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A From a to v to s

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 6]

A particle moves in a straight line. Its acceleration, in m s^{-2}, at time t seconds is $a(t) = 6t - 4$. When $t = 0$, its velocity is $3\ \text{m s}^{-1}$ and its displacement from a fixed point O is 2 m.

(a) Find an expression for the velocity $v(t)$. [3]

(b) Find an expression for the displacement $s(t)$. [2]

(c) Find the displacement of the particle when $t = 2$. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Integrate a, with a constant	$v(t) = 3t^2 - 4t + c$	A1A1
	A1 for each term.	
Step 2 $v(0)$ fixes c	$v(0) = 3 \Rightarrow v(t) = 3t^2 - 4t + 3$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Integrate v, with a constant	$s(t) = t^3 - 2t^2 + 3t + d$	A1
Step 2 $s(0)$ fixes d	$s(0) = 2 \Rightarrow s(t) = t^3 - 2t^2 + 3t + 2$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Substitute $t = 2$	$s(2) = 8 - 8 + 6 + 2 = 8\ \text{m}$	A1

EXAM-STYLE QUESTION 2 Paper 1 · no calculator · [Maximum mark: 5]

The velocity of a particle, in m s^{-1}, is $v(t) = 4\cos(2t)$, for $t \geq 0$. Its initial displacement from O is 1 m.

(a) Find an expression for the displacement $s(t)$. [3]

(b) Find the displacement when $t = \dfrac{\pi}{12}$. [1]

(c) Write down the greatest displacement of the particle from O. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Integrate v, chain rule	$s(t) = 2\sin(2t) + c$ *A1 for* $\sin(2t)$*, A1 for* $\frac{4}{2} = 2$.	A1A1
Step 2 $s(0)$ fixes c	$s(0) = 1 \Rightarrow s(t) = 2\sin(2t) + 1$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Substitute	$2\sin\dfrac{\pi}{6} + 1 = 2$ m	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Greatest value of sin	3 m $\sin(2t)$ *is at most* 1.	A1

B Displacement and distance

EXAM-STYLE QUESTION 3 — Paper 1 · no calculator · [Maximum mark: 8]

A particle moves in a straight line with velocity $v(t) = t^2 - 4t + 3$ m s^{-1}, for $0 \le t \le 4$, where t is in seconds.

(a) Find the displacement of the particle at $t = 4$ from its position at $t = 0$. [3]

(b) Find the times at which the particle is at rest. [2]

(c) Find the total distance travelled by the particle. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The integral, with its limits	$\int_0^4 (t^2 - 4t + 3)\,\mathrm{d}t$	M1
Step 2 Antiderivative	$\left[\frac{t^3}{3} - 2t^2 + 3t\right]_0^4$	A1
Step 3 Substitute the limits	$= \frac{4}{3}$ m	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 At rest: factorise v	$(t-1)(t-3) = 0$	(M1)
Step 2 Solve	$t = 1,\ t = 3$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Integrate on each piece	$\int_0^1 v\,\mathrm{d}t = \frac{4}{3}$, $\int_1^3 v\,\mathrm{d}t = -\frac{4}{3}$, $\int_3^4 v\,\mathrm{d}t = \frac{4}{3}$	M1
	One integral per piece, between the zeros of v.	
Step 2 Add the sizes	$\frac{4}{3} + \frac{4}{3} + \frac{4}{3}$	A1
	Add the sizes.	
Step 3 Distance	$= 4$ m	A1
	$\frac{4}{3}$, the displacement, earns no marks here.	

EXAM-STYLE QUESTION 4

Paper 1 · no calculator · [Maximum mark: 9]

A particle starts from rest at a point O and moves in a straight line. Its acceleration at time t seconds is $a(t) = 12 - 6t$ m s^{-2}.

(a) Find an expression for the velocity $v(t)$. [2]

(b) Find an expression for the displacement $s(t)$ from O. [2]

(c) Find the time, $t > 0$, at which the particle returns to O. [2]

(d) Find the distance travelled by the particle in the first 6 seconds. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Integrate a	$v(t) = 12t - 3t^2 + c$	A1
Step 2 From rest: $v(0) = 0$	from rest: $c = 0$, $v(t) = 12t - 3t^2$	A1

"From rest": $v(0) = 0$.

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Integrate v	$s(t) = 6t^2 - t^3 + d$	A1
Step 2 From O: $s(0) = 0$	from O: $d = 0$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Back at O: $s = 0$	$t^2(6 - t) = 0$	(M1)
Step 2 Solve	$t = 6$ s	A1

WORKING (d)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Turning time: $v = 0$	$v = 0$ at $t = 4$: $s(4) = 96 - 64 = 32$	M1
Step 2 Add the legs	$\lvert s(4) - s(0)\rvert + \lvert s(6) - s(4)\rvert = 32 + 32$	A1
Step 3 Distance	$= 64$ m	A1

The turning time first.

C On the GDC (Paper 2)

EXAM-STYLE QUESTION 5

Paper 2 · calculator · [Maximum mark: 6]

A particle moves in a straight line with velocity $v(t) = 3\sin t + t - 3$ m s^{-1}, for $0 \le t \le 6$, where t is in seconds.

(a) Find the times at which the particle is at rest. [2]

(b) Find the displacement of the particle at $t = 6$ from its position at $t = 0$. [2]

(c) Find the total distance travelled by the particle. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Zeros of v on the GDC	$t = 0.816,\ t = 3.21,\ t = 5.37$	A2
	A1 for two of the three.	

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Displacement: $\int v$	$\displaystyle\int_0^6 (3\sin t + t - 3)\,\mathrm{d}t$	(M1)
Step 2 Compute	$= 0.119$ m $\ (0.119489\ldots)$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Distance: $\int \lvert v\rvert$	$\displaystyle\int_0^6 \lvert 3\sin t + t - 3\rvert\,\mathrm{d}t$	(M1)
Step 2 Compute	$= 6.54$ m $\ (6.54102\ldots)$	A1

EXAM-STYLE QUESTION 6

Paper 2 · calculator · [Maximum mark: 5]

A particle moves in a straight line, starting at O. Its velocity is $v(t) = 8\cos t - t$ m s^{-1}, for $t \ge 0$.

(a) Find an expression for the displacement $s(t)$ of the particle from O. [3]

(b) Find the first time $T > 0$ at which the particle is back at O. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Integrate v	$s(t) = 8\sin t - \dfrac{t^2}{2} + c$	A1A1
	A1 for each term.	
Step 2 $s(0)$ fixes c	$s(0) = 0 \Rightarrow c = 0$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Back at O: $s(T) = 0$	$8\sin T - \dfrac{T^2}{2} = 0$	(M1)
	Or $\int_0^T v(t)\,\mathrm{d}t = 0$.	
Step 2 Solve on the GDC	$T = 2.68$ s $\ (2.67713\ldots)$	A1

V WHICH METHOD? READ THE QUESTION

5.9 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
$a(t)$ *given; "find* $v(t)$*"*	$v(0)$	$v = \int a\,\mathrm{d}t + c$; c from $v(0)$ P1
$v(t)$ *given; "find* $s(t)$*"*	$s(0)$	$s = \int v\,\mathrm{d}t + d$; d from $s(0)$ P1
"starts from rest", "starts at O"	$v(0) = 0$ $s(0) = 0$	each condition fixes its own constant P1
"displacement between t_1 *and* t_2*"*	v	$\int_{t_1}^{t_2} v\,\mathrm{d}t$, with its sign P2
"distance travelled"	the zeros of v	Paper 1: split at the zeros, add the sizes; Paper 2: $\int \lvert v\rvert\,\mathrm{d}t$ P3
"returns to its starting point"	$s(T) = s(0)$	solve $\int_0^T v\,\mathrm{d}t = 0$ for $T > 0$ P4
Paper 2, v *with* sin, e *or* ln	the GDC	zeros of v; $\int v$; $\int \lvert v\rvert$; each integral written in full P4

In the exam 1 *command terms*

From rest: $v = 0$. **At** O**, at the origin**: $s = 0$. **Initially**: $t = 0$. **Displacement**: signed. **Distance travelled**: always positive, every leg counted. Units: m, $\mathrm{m\,s^{-1}}$, $\mathrm{m\,s^{-2}}$.

VI ON THE GDC: PAPER 2

SL 5.9 · 5.11 · NumWorks · every calculator move of this topic

1 When is the particle at rest?

Grapher → v(x) → Find → Zeros

v with x for t, the window on the time interval, then each zero. **On paper:** "$v(t) = 0$", then the times.

GRAPHER · ZEROS

```
f(x)=3sin(x)+x-3
x=0.815596 y=0
x=3.212478 y=0
x=5.371539 y=0
```

2 Displacement and distance

Calculation → Integral → v, then abs(v)

The same integral twice: once with v (displacement), once with $|v|$ (distance).

CALCULATION

```
int(3sin(x)+x-3,x,0,6)
0.1194891400
int(abs(3sin(x)+x-3),x,0,6)
6.541028290
```

ANSWERS TO THE APPLY QUESTIONS

I.1 $8t - t^2$

I.2 $3\mathrm{e}^{2t} - 2$

I.3 $10t - 2t^2$

I.4 $3\sin t - 2$

I.5 $t = 2;\ s(2) = 8$ m

I.6 $v = 6t^2,\ s = 2t^3;\ 16$ m

II.1 10 m

II.2 0: back at the start

II.3 $\frac{9}{2} + 2 = \frac{13}{2}$ m; $-\frac{5}{2}$ m

II.4 $2 + 4 = 6$ m; 2 m

III.1 1.57 s; -0.305 m; 1.45 m

III.2 $s = 10 - 10\mathrm{e}^{-t} - 2t;\ T = 4.97$ s

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