Exponents and logarithms

Exponents turn repeated multiplication into a single number; logarithms go back, from the result to the exponent. Their laws mirror each other: what is a product for powers becomes a sum for logarithms. With them, an unknown in an exponent can be brought down and solved for.

I LAWS OF EXPONENTS

SL 1.5 · 1.7 · integer and rational exponents

I.1 The laws of exponents

Property 1 *the laws of exponents*

For $a > 0$: $a^m \times a^n = a^{m+n}$, $\dfrac{a^m}{a^n} = a^{m-n}$, $(a^m)^n = a^{mn}$, $(ab)^n = a^n b^n$.

Watch out 1

$a^m \times a^n$ is not a^{mn}, and $a^m + a^n$ cannot be simplified.

Example 1

Simplify $(3x^2)^3 \times x^{-4}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Power of a product: each factor	$3^3 (x^2)^3 = 27x^6$
Step 2 Multiply: add the exponents	$27x^6 \times x^{-4} = 27x^{6-4}$
Step 3 Simplify	$27x^2$

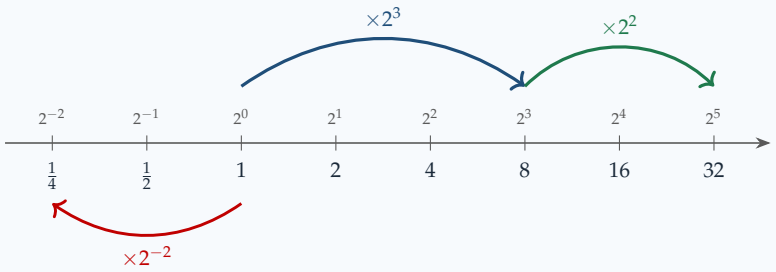


Figure 1 — Powers of 2, each step ×2: $2^3 \times 2^2$ is three steps then two more, $2^5 = 32$. Steps to the left divide: $2^{-2} = \frac{1}{4}$.

APPLY

1 Simplify $x^5 \times x^{-2}$.

2 Simplify $(2a^3)^4$.

3 Simplify $\dfrac{12y^7}{4y^2}$.

I.2 Zero, negative and fractional exponents

Definition 1 *zero, negative and rational exponents*

$$a^0 = 1,\ a^{-n} = \frac{1}{a^n},\ a^{\frac{1}{n}} = \sqrt[n]{a},\ a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m.$$

Example 2

Find the value of $625^{\frac{3}{4}}$ and of $25^{-\frac{1}{2}}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 The denominator is a root: take it first	$\sqrt[4]{625} = 5,\ \sqrt{25} = 5$
Step 2 The numerator is a power	$625^{\frac{3}{4}} = 5^3 = 125$
Step 3 A minus sign: the reciprocal	$25^{-\frac{1}{2}} = \frac{1}{5}$

APPLY

4 $81^{\frac{1}{4}}$

5 $8^{-\frac{2}{3}}$

6 $\left(\frac{9}{4}\right)^{\frac{3}{2}}$

7 7^0

I.3 Equations with the same base

Example 3

Solve $9^x = 27^{x-1}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Write both sides as powers of 3	$3^{2x} = 3^{3(x-1)}$
Step 2 Same base: equate the exponents	$2x = 3x - 3$
Step 3 Solve	$x = 3$

APPLY

8 Solve $2^{x+3} = 8^x$.

9 Solve $5^{2x-1} = 125$.

10 Solve $4^x = \frac{1}{8}$.

II LOGARITHMS

SL 1.5 · 1.7 · $\log_a x$, the laws, change of base, exponential equations

II.1 The logarithm is an exponent

Definition 2 *logarithm*

For $a > 0$, $a \neq 1$, $x > 0$: $\log_a x = y \iff a^y = x$. $\log x$ is $\log_{10} x$; $\ln x$ is $\log_e x$. Always $\log_a 1 = 0$ and $\log_a a = 1$.

Example 4

Find $\log_5 125$, $\log 0.0001$ and $\log_9 3$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Ask: the base to what power?	$5^? = 125,\ 10^? = 0.0001,\ 9^? = 3$
Step 2 Read the exponent	$3,\ -4,\ \frac{1}{2}$

APPLY

1 $\log_2 \frac{1}{8}$

2 $\log_7 49$

3 $\ln \sqrt{e}$

II.2 The laws of logarithms

Property 2 *the laws of logarithms*

$\log_a(xy) = \log_a x + \log_a y,\ \log_a \frac{x}{y} = \log_a x - \log_a y,\ \log_a x^m = m \log_a x$; change of base $\log_a x = \frac{\ln x}{\ln a}$.

Watch out 2

$\log(x + y)$ is not $\log x + \log y$, and $\frac{\log x}{\log y}$ is not $\log \frac{x}{y}$.

Example 5

Find the value of $2 \log 5 + \log 4$, without a calculator.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Power law first	$2 \log 5 = \log 25$
Step 2 Product law: one logarithm	$\log 25 + \log 4 = \log 100$
Step 3 Evaluate	$\log 100 = 2$

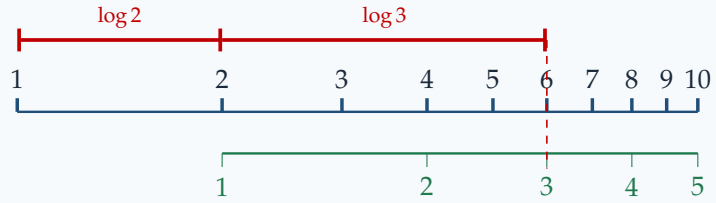


Figure 2 — A slide rule: the distance from 1 to x is $\log x$. The length $\log 2$ placed after $\log 3$ reaches $\log 6$: $\log(2 \times 3) = \log 2 + \log 3$.

APPLY

4 $\log_6 9 + \log_6 4$

5 $\log_2 40 - \log_2 5$

6 $3\log_4 2 + \log_4 2$

II.3 In terms of p and q

Example 6

Let $\log_2 a = p$ and $\log_2 b = q$. Write $\log_2 \dfrac{8a}{b^3}$ in terms of p and q.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Split the quotient and the product	$\log_2 8 + \log_2 a - \log_2 b^3$
Step 2 Power law	$\log_2 8 + \log_2 a - 3\log_2 b$
Step 3 Replace, evaluate $\log_2 8$	$3 + p - 3q$

APPLY

7 $\ln a = p$, $\ln b = q$: write $\ln\left(a^2 b\right)$.

8 $\log a = p$: write $\log \dfrac{\sqrt{a}}{100}$.

II.4 Solving $a^x = b$

Example 7

Solve $2 \times 3^x = 30$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Isolate the power	$3^x = 15$
Step 2 Take logarithms	$x \ln 3 = \ln 15$
Step 3 Divide, then compute	$x = \dfrac{\ln 15}{\ln 3} = 2.46$

APPLY

9 Solve $6^x = 100$.

10 Solve $1.05^n = 3$.

II.5 Solving a logarithmic equation

Example 8

Solve $\log_3(x+1) + \log_3(x-1) = 1$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 One logarithm: the product law	$\log_3\big((x+1)(x-1)\big) = 1$
Step 2 Undo the logarithm	$x^2 - 1 = 3^1 = 3$
Step 3 Solve	$x = 2$ or $x = -2$
Step 4 Reject what makes an argument ≤ 0	$x = -2$ gives $\log_3(-1)$: $x = 2$

APPLY

11 Solve $\log_2 x - \log_2(x-3) = 2$.

12 Solve $\ln(2x+3) = \ln x + \ln 3$.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A Laws of exponents

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 5]

Answer the following without a calculator.

(a) Find the value of $27^{\frac{2}{3}}$. [1]

(b) Find the value of $\left(\frac{1}{4}\right)^{-\frac{3}{2}}$. [1]

(c) Write $4^x \times 8^{x-1}$ as a power of 2. [2]

(d) Hence solve $4^x \times 8^{x-1} = 128$. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Root first, then the power	9	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 A minus sign: the reciprocal, then root and power	8	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Write both as powers of 2	$2^{2x} \times 2^{3(x-1)}$	(M1)
Step 2 Add the exponents	$= 2^{5x-3}$	A1

WORKING (d)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Same base: equate the exponents, solve	$5x - 3 = 7,\ x = 2$	A1

$128 = 2^7$.

B A logarithmic equation

EXAM-STYLE QUESTION 2 Paper 1 · no calculator · [Maximum mark: 5]

Solve the equation $\log_2 x + \log_2(x-2) = 3$, for $x > 2$.

(a) Show that $x^2 - 2x - 8 = 0$. [2]

(b) Hence find the value of x, justifying your answer. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 One logarithm: the product law	$\log_2\left(x(x-2)\right) = 3$	M1
Step 2 Undo the logarithm	$x(x-2) = 2^3 = 8$	A1
Step 3 Rearrange	$x^2 - 2x - 8 = 0$	AG

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Factorise	$(x-4)(x+2) = 0$	(M1)
Step 2 Solve	$x = 4$	A1
Step 3 Reject what makes an argument ≤ 0	$x = -2$ rejected: $\log_2(-2)$ is not defined	R1
	Or: $x > 2$.	

C The laws of logarithms

EXAM-STYLE QUESTION 3 — Paper 1 · no calculator · [Maximum mark: 3]

Let $\log_3 a = p$ and $\log_3 b = q$.

(a) Write $\log_3\left(\frac{9a^2}{b}\right)$ in terms of p and q. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Split the quotient, product and power	$\log_3 9 + 2\log_3 a - \log_3 b$	(M1)
Step 2 Evaluate $\log_3 9$	$\log_3 9 = 2$	(A1)
Step 3 Replace by p and q	$2 + 2p - q$	A1

D With the GDC (Paper 2)

EXAM-STYLE QUESTION 4 — Paper 2 · calculator · [Maximum mark: 5]

Answer the following.

(a) Solve $3^{x+1} = 5^x$. [3]

The population of a town t years after 2020 is modelled by $P = 1200 \times 1.04^t$.

(b) Find the number of years after 2020 when the population first reaches 2000. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Take logarithms	$(x+1)\ln 3 = x \ln 5$	M1
	Or the GDC.	
Step 2 Collect the x terms, factorise	$x = \dfrac{\ln 3}{\ln 5 - \ln 3}$	(A1)
Step 3 Compute	$x = 2.15 \ \ (2.15066\ldots)$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Write the equation	$1200 \times 1.04^t = 2000$	(M1)
Step 2 Solve on the GDC, the first year	$t = 13.0$ years $\ \ (13.0243\ldots)$	A1
	Accept 14 years if the answer is given in whole years with reason.	

IV WHICH METHOD? READ THE QUESTION

1.7 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"simplify", "as a power of 2"	the same base	$4 = 2^2$, $8 = 2^3$, then the laws P1
a fractional or negative exponent	$a^{\frac{m}{n}}$, a^{-n}	root first, then the power; $-$: the reciprocal D1
an unknown in the exponent	logarithms	same base: equate exponents; otherwise $x = \frac{\ln b}{\ln a}$ P2
a sum of logarithms $= k$	one logarithm	$\log_a(xy) = k$, then $xy = a^k$ P2
"justify", "reject"	the argument > 0	check every solution in the original equation W2
"in terms of p and q"	the laws, backwards	split the product, quotient and powers P2
a base the GDC does not have	change of base	$\log_a x = \frac{\ln x}{\ln a}$ P2

In the exam 1 *command terms*

Exact: leave logarithms and roots, no decimals. **Hence**: use the result just shown. **Justify**: the reason a value is kept or rejected, in words.

V ON THE GDC: PAPER 2

SL 1.5 · 1.7 · NumWorks · every calculator move of this topic

1 An exponential equation

Equations → Add an equation → Solve

The equation solver, or the intersection of $y = 3^{x+1}$ and $y = 5^x$. **On paper:** $(x+1)\ln 3 = x \ln 5$, then $x = 2.15$.

EQUATIONS · SOLUTION

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3^(x+1)=5^x
x = 2.150660
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ANSWERS TO THE APPLY QUESTIONS

I.1 x^3

I.2 $16a^{12}$

I.3 $3y^5$

I.4 3

I.5 $\frac{1}{4}$

I.6 $\frac{27}{8}$

I.7 1

I.8 $x = \frac{3}{2}$

I.9 $x = 2$

I.10 $x = -\frac{3}{2}$

II.1 -3

II.2 2

II.3 $\frac{1}{2}$

II.4 2

II.5 3

II.6 2

II.7 $2p + q$

II.8 $\frac{p}{2} - 2$

II.9 2.57

II.10 22.5

II.11 $x = 4$

II.12 $x = 3$

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