Geometric sequences and series

A geometric sequence multiplies by the same number at every step. With a ratio above 1 it explodes; with a ratio between -1 and 1 it shrinks towards 0, so fast that infinitely many terms can add up to a finite number.

I GEOMETRIC SEQUENCES AND SERIES

SL 1.3 · $u_n = u_1 r^{n-1}$, S_n, the common ratio

I.1 The common ratio and u_n

Definition 1 *geometric sequence*

Each term is the previous one multiplied by the same number r, the **common ratio**: $r = \dfrac{u_{n+1}}{u_n}$. Then $u_n = u_1\, r^{\,n-1}$.

Watch out 1

Divide, do not subtract, to find r. The power is $n-1$. And $(-2)^{n-1}$ is not -2^{n-1}: keep the brackets for a negative ratio.

Example 1

Consider the sequence $5,\ 15,\ 45,\ \ldots$ Find the common ratio, a formula for u_n, and u_7.

WHAT YOU DO	WHAT YOU WRITE
Step 1 r: divide consecutive terms	$r = \dfrac{15}{5} = 3$
Step 2 Write $u_1 r^{n-1}$	$u_n = 5 \times 3^{n-1}$
Step 3 Substitute $n = 7$	$u_7 = 5 \times 3^6 = 3645$

APPLY

1 $80,\ 40,\ 20,\ \ldots$: find r and u_6.

2 $3,\ -6,\ 12,\ \ldots$: find u_6.

3 $u_n = 4 \times 2^{n-1}$: first term and common ratio.

I.2 From two terms

Property 1 *two terms are enough*

Between u_k and u_m there are $m-k$ steps of $\times r$: $\dfrac{u_m}{u_k} = r^{m-k}$.

Example 2

In a geometric sequence, $u_3 = 50$ and $u_6 = 3.2$. Find r and u_1.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Divide the terms: 3 steps	$r^3 = \dfrac{3.2}{50} = 0.064$
Step 2 Take the cube root	$r = 0.4$
Step 3 Go back to u_1: 2 steps	$u_1 = \dfrac{50}{0.4^2} = 312.5$

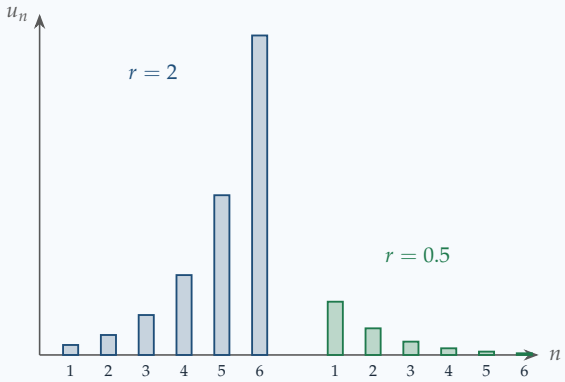


Figure 1 — The first terms of $u_n = 6 \times 2^{n-1}$ (Exam-style question 1) grow faster and faster; those of $u_n = 32 \times 0.5^{n-1}$ halve at each step.

APPLY

4 $u_2 = 6$, $u_4 = 54$, $r > 0$: find r and u_1.

5 $u_1 = 7$, $u_4 = -56$: find r.

I.3 The sum S_n

Note 1 *the sum of n terms*

$S_n = \dfrac{u_1(r^n - 1)}{r - 1} = \dfrac{u_1(1 - r^n)}{1 - r}$, for $r \neq 1$: the first form when $r > 1$, the second when $r < 1$.

Example 3

Find the sum of the first 6 terms of $4,\ 12,\ 36,\ \ldots$

WHAT YOU DO	WHAT YOU WRITE
Step 1 Read u_1, r, n	$u_1 = 4, r = 3, n = 6$
Step 2 Substitute in S_n	$S_6 = \dfrac{4(3^6 - 1)}{3 - 1}$
Step 3 Compute	$= 2 \times 728 = 1456$

APPLY

6 $1 + 2 + 4 + \cdots$: find S_{12}.

7 $81 + 27 + 9 + \cdots$: find S_5.

I.4 Finding n

Example 4

Find the least n such that the sum of the first n terms of $5,\ 10,\ 20,\ \ldots$ is more than 10 000.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Write S_n with n	$5\,(2^n - 1) > 10\,000$
Step 2 Solve on the GDC (or with logarithms)	$2^n > 2001$: $n > 10.97$
Step 3 Next integer up, and check	$S_{10} = 5115$, $S_{11} = 10\,235$: $n = 11$

APPLY

8 $u_n = 3 \times 2^{n-1}$: least n with $u_n > 1000$.

9 $u_n = 100 \times 0.9^{n-1}$: least n with $u_n < 1$.

II INFINITE GEOMETRIC SERIES

SL 1.8 · $S_\infty = \frac{u_1}{1-r}$ when $|r| < 1$

II.1 The sum to infinity

Definition 2 *sum to infinity*

When $|r| < 1$, S_n tends to the **sum to infinity** $S_\infty = \dfrac{u_1}{1-r}$. When $|r| \geq 1$, there is none: the series **diverges**.

Example 5

Find the sum to infinity of $40 + 10 + 2.5 + \cdots$

WHAT YOU DO	WHAT YOU WRITE
Step 1 Find r	$r = \dfrac{10}{40} = \dfrac{1}{4}$
Step 2 Say why S_∞ exists	$\lvert r\rvert < 1$
Step 3 Use $\dfrac{u_1}{1-r}$	$S_\infty = \dfrac{40}{1-\frac{1}{4}} = \dfrac{160}{3} = 53.3$

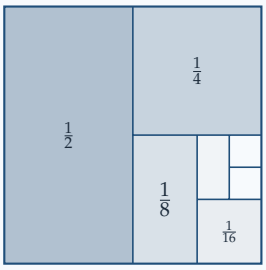


Figure 2 — $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$: each term fills half of what is left of the unit square, $S_\infty = \dfrac{1/2}{1-1/2} = 1$.

APPLY

1 $6 - 3 + 1.5 - \cdots$: find S_∞.

2 $5 + 7.5 + 11.25 + \cdots$: find S_∞.

II.2 Finding u_1 or r from S_∞

Example 6

A geometric series has first term 12 and sum to infinity 30. Find r.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Write the equation	$\dfrac{12}{1-r} = 30$
Step 2 Solve for $1-r$	$1-r = 0.4$
Step 3 Conclude, check $\lvert r\rvert < 1$	$r = 0.6$

APPLY

3 $S_\infty = 50$, $r = 0.8$: find u_1.

4 $S_\infty = 24$, $u_1 = 18$: find r.

II.3 A condition on x

Property 2 *a condition on x*

When the ratio depends on x, the sum to infinity exists exactly when $\lvert r\rvert < 1$, that is $-1 < r < 1$.

Watch out 2

Before using S_∞, state why it exists: "$\lvert r\rvert < 1$". A sum to infinity is a limit: no partial sum ever equals it.

Example 7

Consider the series $3 + 3(x+1) + 3(x+1)^2 + \cdots$ Find the values of x for which it has a sum to infinity, and that sum.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Find r	$r = x+1$
Step 2 Write $-1 < r < 1$	$-1 < x+1 < 1$
Step 3 Solve	$-2 < x < 0$
Step 4 Use $\dfrac{u_1}{1-r}$	$S_\infty = \dfrac{3}{1-(x+1)} = -\dfrac{3}{x}$

APPLY

5 $1 + 3x + 9x^2 + \cdots$: for which x is there a sum to infinity?

6 $1 + (x-2) + (x-2)^2 + \cdots$: for which x, and the sum?

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A Geometric sequences

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 5]

In a geometric sequence, $u_2 = 12$ and $u_5 = 96$.

(a) Find the common ratio. [2]

(b) Find the first term. [1]

(c) Find the sum of the first 6 terms. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Divide the terms: 3 steps	$r^3 = \dfrac{96}{12} = 8$	(M1)
Step 2 Take the cube root	$r = 2$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Go back to u_1	$u_1 = \dfrac{12}{2} = 6$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Substitute in S_n	$S_6 = \dfrac{6\left(2^6 - 1\right)}{2 - 1}$	(M1)
Step 2 Compute	$= 378$	A1

B The sum to infinity

EXAM-STYLE QUESTION 2

Paper 1 · no calculator · [Maximum mark: 4]

The first two terms of a geometric series are 24 and 18.

(a) Find the common ratio. [1]

(b) Explain why the series has a sum to infinity. [1]

(c) Find the sum to infinity. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Divide consecutive terms	$r = \frac{18}{24} = \frac{3}{4}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Say why S_∞ exists	$\lvert r \rvert < 1$	R1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Use $\frac{u_1}{1-r}$	$S_\infty = \frac{24}{1-\frac{3}{4}}$	(M1)
Step 2 Compute	$= 96$	A1

EXAM-STYLE QUESTION 3

Paper 1 · no calculator · [Maximum mark: 4]

Consider the geometric series $1 + (2x-1) + (2x-1)^2 + \cdots$.

(a) Find the values of x for which the series has a sum to infinity. [2]

(b) Find the sum to infinity when $x = 0.25$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Write $-1 < r < 1$	$-1 < 2x - 1 < 1$	(M1)
Step 2 Solve	$0 < x < 1$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Find r, use $\frac{u_1}{1-r}$	$r = -0.5,\ \frac{1}{1-(-0.5)}$	(M1)
Step 2 Compute	$= \frac{2}{3}$	A1

C In context, with the GDC (Paper 2)

EXAM-STYLE QUESTION 4 Paper 2 · calculator · [Maximum mark: 7]

A ball is dropped from a height of 2 m. After each bounce, it rises to 80 % of the height from which it last fell.

(a) Find the height the ball reaches after the third bounce. [2]

(b) Find the total vertical distance travelled by the ball before it comes to rest. [3]

(c) Find the number of bounces after which the ball first rises to less than 0.1 m. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Height after 3 bounces: 2×0.8^3	2×0.8^3	(M1)
Step 2 Compute	$= 1.024$ m $\quad (1.02)$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Up and down after the first fall: a sum to infinity	up and down after the first fall: $2 \times \dfrac{1.6}{1-0.8}$	M1
	The heights after the bounces: $1.6,\ 1.28,\ \ldots$*, each travelled twice.*	
Step 2 Compute	$= 16$	(A1)
Step 3 Add the first fall	$2 + 16 = 18$ m	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Write the inequality	$2 \times 0.8^n < 0.1$	(M1)
Step 2 Solve on the GDC, next integer	$n = 14$	A1
	$n > 13.4$.	

IV WHICH METHOD? READ THE QUESTION

1.8 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"geometric", "common ratio"	$u_n = u_1 r^{n-1}$	r by dividing two consecutive terms D1
two terms given	a power of r	$\frac{u_m}{u_k} = r^{m-k}$ P1
"the sum of the first n terms"	S_n	$\frac{u_1(r^n - 1)}{r - 1}$ N1
"sum to infinity"	$\lvert r \rvert < 1$	say it, then $\frac{u_1}{1 - r}$ D2
"for which x does it exist?"	$\lvert r \rvert < 1$	a double inequality in x P2
"the least n such that ..."	an inequality with r^n	logarithms or the GDC table; the next integer
a negative ratio	brackets	$(-2)^{n-1}$, terms alternate in sign W1

In the exam 1 *command terms*

Explain: a reason in words, here "$|r| < 1$". **Find**: working, then the answer. **Exact**: a fraction, not a decimal, on Paper 1.

V ON THE GDC: PAPER 2

SL 1.3 · 1.8 · NumWorks · every calculator move of this topic

1 Terms and sums of a geometric sequence

Sequences → Explicit → u(n) → Table

The table shows the first n for which a term falls below a value. **On paper:** $2 \times 0.8^{14} = 0.0880 < 0.1$, $2 \times 0.8^{13} = 0.110$, so $n = 14$.

SEQUENCES · TABLE

```
u(n)=2*0.8^n
n u(n)
13 0.10995
14 0.08796
```

ANSWERS TO THE APPLY QUESTIONS

I.1 $\frac{1}{2}$; 2.5

I.2 $3 \times (-2)^5 = -96$

I.3 4; 2

I.4 3; 2

I.5 -2

I.6 4095

I.7 121

I.8 $n = 10$

I.9 $n = 45$

II.1 $r = -\frac{1}{2}$; 4

II.2 none: $r = 1.5$, $|r| \geq 1$

II.3 10

II.4 $\frac{1}{4}$

II.5 $-\frac{1}{3} < x < \frac{1}{3}$

II.6 $1 < x < 3;\ \dfrac{1}{3-x}$

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