The binomial theorem

Expanding $(a+b)^n$ by hand means multiplying n brackets. The binomial theorem gives the answer directly: every term is a power of a times a power of b, and the numbers in front are the entries of Pascal's triangle, $\binom{n}{r}$. Most questions ask for one term only.

I PASCAL'S TRIANGLE AND COEFFICIENTS

SL 1.9 · $\binom{n}{r} = {}^nC_r$, Pascal's triangle, $n!$

I.1 $n!$ and $\binom{n}{r}$

Definition 1 *binomial coefficient*

$\binom{n}{r} = {}^nC_r = \dfrac{n!}{r!\,(n-r)!}$, with $n! = 1 \times 2 \times \cdots \times n$ and $0! = 1$: the number of ways of choosing r objects from n. $\binom{n}{0} = \binom{n}{n} = 1$ and $\binom{n}{r} = \binom{n}{n-r}$.

Example 1

Find $\binom{8}{3}$ without a calculator.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Write the formula	$\dfrac{8!}{3!\,5!}$
Step 2 Cancel 5!	$\dfrac{8 \times 7 \times 6}{3 \times 2 \times 1}$
Step 3 Compute	56

APPLY

1 $\binom{9}{2}$

2 $\binom{10}{8}$

3 $4!$

I.2 Pascal's triangle

Property 1 *Pascal's triangle*

Row n lists $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$. Each entry is the sum of the two above it. Row n adds up to 2^n.

Watch out 1

Rows start at row 0: 1 4 6 4 1 is row 4, the fifth line. And $\binom{n}{2} = \dfrac{n(n-1)}{2}$: useful when n is the unknown.

Example 2

Row 5 of Pascal's triangle is 1 5 10 10 5 1. Write down row 6.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Start and end with 1	$1 \ldots 1$
Step 2 Add each pair of neighbours	$1+5, 5+10, 10+10, 10+5, 5+1$
Step 3 Write the row, check the sum 2^6	1 6 15 20 15 6 1, sum 64

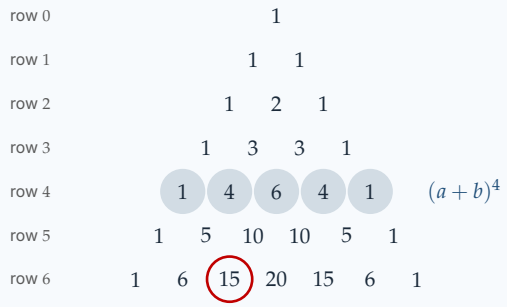


Figure 1 — Rows 0 to 6 of Pascal's triangle; row 6 is Example 2. Row 4, 1 4 6 4 1, gives the coefficients of $(a+b)^4$.

APPLY

4 The sum of row 7.

5 Solve $\binom{n}{2} = 45$.

II.1 Expanding $(a+b)^n$

Property 2 *the binomial theorem*

$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r$. The **general term** is $\binom{n}{r}a^{n-r}b^r$: the powers of a and b add up to n.

Example 3

Expand $(x-3)^3$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Coefficients: row 3	1 3 3 1
Step 2 Powers of x down, powers of (-3) up	$x^3 + 3x^2(-3) + 3x(-3)^2 + (-3)^3$
Step 3 Simplify each term	$x^3 - 9x^2 + 27x - 27$

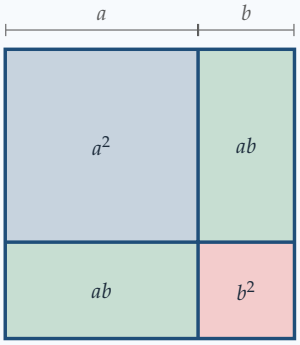


Figure 2 — $(a+b)^2 = a^2 + 2ab + b^2$ as areas: two squares and two equal rectangles — the coefficients 1, 2, 1 of row 2.

APPLY

1 Expand $(1+x)^4$.

2 Expand $(2x+1)^3$.

II.2 One term only

Property 3 *one term only*

Write the general term with each part in brackets, simplify the power of x, and choose r.

Watch out 2

$(4x)^3 = 64x^3$, not $4x^3$, and $(-1)^2 = 1$. The coefficient is the number only; the term includes the power of x.

Example 4

Find the term in x^3 in the expansion of $(4x - 1)^5$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 General term, parts in brackets	$\binom{5}{r}(4x)^{5-r}(-1)^r$
Step 2 Choose r for x^3	$5 - r = 3$: $r = 2$
Step 3 Substitute and simplify	$10 \times 64x^3 \times 1 = 640x^3$

APPLY

3 The coefficient of x^2 in $(1 + 3x)^6$.

4 The term in x^4 in $(x - 2)^6$.

II.3 The constant term

Example 5

Find the constant term in the expansion of $\left(x - \frac{2}{x}\right)^6$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 General term	$\binom{6}{r}x^{6-r}\left(-\frac{2}{x}\right)^r = \binom{6}{r}(-2)^r x^{6-2r}$
Step 2 Power of x equal to 0	$6 - 2r = 0$: $r = 3$
Step 3 Substitute	$20 \times (-8) = -160$

APPLY

5 The constant term of $\left(x^2 + \frac{1}{x}\right)^6$.

6 The constant term of $\left(2x + \frac{1}{x}\right)^4$.

II.4 An unknown in the bracket

Example 6

The coefficient of x^2 in the expansion of $(1+ax)^5$ is 90, and $a>0$. Find a.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Write the term in x^2	$\binom{5}{2}(ax)^2 = 10a^2x^2$
Step 2 Equate the coefficient	$10a^2 = 90$
Step 3 Solve, keep $a>0$	$a = 3$

APPLY

7 The coefficient of x in $(2+kx)^4$ is 96. Find k.

II.5 A product of two brackets

Example 7

Find the coefficient of x^2 in the expansion of $(1+2x)(1+x)^4$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Terms of $(1+x)^4$ needed	$1+4x+6x^2+\ldots$
Step 2 Two ways to make x^2	$1\times 6x^2$ and $2x\times 4x$
Step 3 Add	$6+8=14$

APPLY

8 The coefficient of x in $(3-x)(1+2x)^5$.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A Coefficients

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 3]

Answer the following without a calculator.

(a) Find the value of $\binom{6}{2}$. [1]

(b) Find n, given that $\binom{n}{2} = 28$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Read the coefficient from row 6	15	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $\binom{n}{2} = \dfrac{n(n-1)}{2}$, equate	$\dfrac{n(n-1)}{2} = 28,\ n(n-1) = 56$	(M1)
Step 2 Solve, keep the positive integer	$n = 8$	A1

$n = -7$ is rejected.

B Expansions

EXAM-STYLE QUESTION 2

Paper 1 · no calculator · [Maximum mark: 5]

Consider the expansion of $(2+x)^4$.

(a) Expand $(2+x)^4$ in ascending powers of x, simplifying each term. [3]

(b) Hence find the coefficient of x^2 in the expansion of $(2+x)^4(1-x)$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Coefficients: row 4	coefficients 1, 4, 6, 4, 1	(M1)
Step 2 Write every term, simplified	$16+32x+24x^2+8x^3+x^4$	A2
	A1 for three correct terms.	

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Two ways to make x^2	$24\times 1+32\times(-1)$	(M1)
	The two ways of making x^2.	
Step 2 Add	$=-8$	A1

EXAM-STYLE QUESTION 3

Paper 1 · no calculator · [Maximum mark: 4]

Consider the expansion of $(3x-2)^5$.

(a) Find the term in x^3. [3]

(b) Write down the constant term. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 General term, parts in brackets, $r=2$	$\binom{5}{2}(3x)^3(-2)^2$	(M1)
Step 2 Simplify the numbers	$10\times 27\times 4$	(A1)
Step 3 Write the term	$1080x^3$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The term without x: $r=5$	$(-2)^5=-32$	A1

C A constant term (Paper 2)

EXAM-STYLE QUESTION 4 — Paper 2 · calculator · [Maximum mark: 5]

Consider the expansion of $\left(x^2 + \frac{2}{x}\right)^6$.

(a) Write down the general term, simplifying the power of x. [2]

(b) Find the constant term. [2]

(c) Find the coefficient of x^3. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 General term, parts in brackets	$\binom{6}{r}\left(x^2\right)^{6-r}\left(\frac{2}{x}\right)^r$	(M1)
Step 2 Simplify the power of x	$= \binom{6}{r}2^r x^{12-3r}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Power of x equal to 0	$12 - 3r = 0,\ r = 4$	(M1)
Step 2 Substitute	$\binom{6}{4} \times 2^4 = 240$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Power 3: $r = 3$	$r = 3$: $\binom{6}{3} \times 2^3 = 160$	A1

IV WHICH METHOD? READ THE QUESTION

1.9 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"expand $(a+b)^n$"	row n of Pascal	every term, simplified, in order P1
"find $\binom{n}{r}$", "${}^nC_r$"	$\frac{n!}{r!(n-r)!}$	by hand for small n, otherwise the GDC D1
n unknown in $\binom{n}{2}$	$\frac{n(n-1)}{2}$	a quadratic, keep the positive integer W1
"the term in x^k"	the general term	brackets, the power of x, then r P2
"the coefficient of ..."	the number only	no x in the answer W2
"the constant term"	the power of x is 0	with $\frac{1}{x}$: solve for r P2
a product $(\ldots)^n(1-x)$	two ways to make the power	add the two products P2

In the exam 1 *command terms*

Expand: write every term. **Term**: the coefficient with the power, $1080x^3$. **Coefficient**: the number, 1080. **Hence**: use the expansion just found.

V ON THE GDC: PAPER 2

SL 1.9 · NumWorks · every calculator move of this topic

1 nCr on the GDC

Calculation → Toolbox → Probability → binomial(n,k)

The NumWorks calls $\binom{n}{r}$ `binomial(n,k)`. **On paper:** the general term first, then the value.

CALCULATION

```
binomial(6,4)*2^4
240
binomial(6,3)*2^3
160
```

ANSWERS TO THE APPLY QUESTIONS

I.1 36

I.2 $\binom{10}{2} = 45$

I.3 24

I.4 128

I.5 $n(n-1) = 90$: $n = 10$

II.1 $1 + 4x + 6x^2 + 4x^3 + x^4$

II.2 $8x^3 + 12x^2 + 6x + 1$

II.3 135

II.4 $60x^4$

II.5 15

II.6 24

II.7 $32k = 96$: $k = 3$

II.8 $3 \times 10 - 1 = 29$

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