Inequalities and modulus graphs

HL AA · Polynomials and rational functions · AHL 2.15, 2.16

Solving $f(x) \geq g(x)$ means finding where one graph is above the other. Algebra finds where they are equal and where a denominator is zero; the sign of $f(x) - g(x)$ between those values does the rest. The modulus $|f(x)|$ reflects the parts of a graph below the x-axis, and $f(|x|)$ copies the right half of a graph onto the left.

I POLYNOMIAL AND RATIONAL INEQUALITIES

AHL 2.15 · $g(x) \geq f(x)$: critical values, sign table, or the GDC

I.1 Polynomial inequalities

Property 1 *critical values and signs*

To solve $f(x) \geq g(x)$, bring everything to one side and **factorise**: $h(x) = f(x) - g(x) \geq 0$. The **critical values** are the zeros of h and the zeros of any denominator. Between them h keeps one sign: test one value in each interval, or sketch.

Example 1

Solve $x^3 - 3x^2 - 10x \geq 0$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Factorise	$x(x+2)(x-5) \geq 0$
Step 2 Critical values	$-2,\ 0,\ 5$
Step 3 Sign in each interval	$-$ on $x < -2$; $+$ on $(-2, 0)$; $-$ on $(0, 5)$; $+$ on $x > 5$
Step 4 Write the solution	$-2 \leq x \leq 0$ or $x \geq 5$

APPLY

1 $x^2 < 2x + 8$.

2 $(x-1)^2(x+3) > 0$.

I.2 Rational inequalities

Example 2

Solve $\dfrac{x+5}{x-1} \leq 2$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 One side zero, one fraction	$\dfrac{x+5-2(x-1)}{x-1} \leq 0$: $\dfrac{7-x}{x-1} \leq 0$
Step 2 Critical values	7 (numerator), 1 (denominator, excluded)
Step 3 Sign in each interval	$-$ on $x<1$; $+$ on $(1,7)$; $-$ on $x>7$
Step 4 Write the solution	$x<1$ or $x \geq 7$

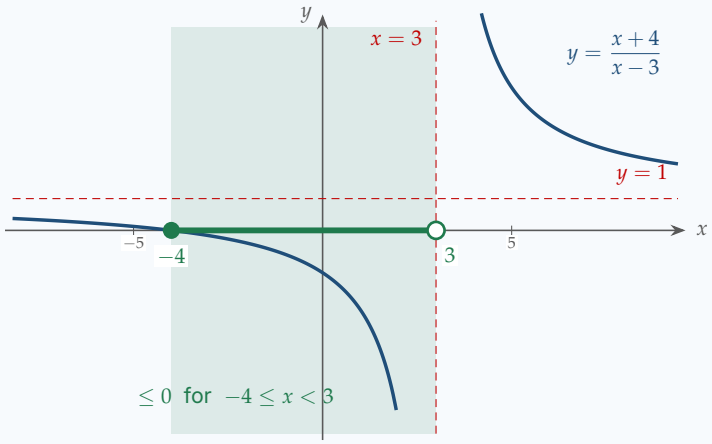


Figure 1 — Exam-style question 1 (a), $\frac{x+4}{x-3} \leq 0$: critical values -4 (numerator) and 3 (denominator); the graph is on or below the axis for $-4 \leq x < 3$ — 3 is excluded, the fraction is undefined there.

Watch out 1

Never multiply an inequality by an expression that may be negative: $\frac{2x+1}{x-3} \leq 1$ is **not** $2x+1 \leq x-3$. Subtract 1 and combine into one fraction, or multiply by $(x-3)^2 > 0$. And never divide by x: $x^3 > 4x$ loses solutions if you do.

APPLY

3 $\dfrac{3}{x+2} > 1$.

4 $\dfrac{x}{x-4} \geq 0$.

II MODULUS GRAPHS, EQUATIONS AND INEQUALITIES

AHL 2.16 · $y = |f(x)|$, $y = f(|x|)$, $|f(x)| = g(x)$

II.1 The two modulus graphs

Definition 1 *the two modulus graphs*

$y = |f(x)|$: keep the parts of $y = f(x)$ on or above the x-axis, **reflect** the parts below it in the x-axis. $y = f(|x|)$: keep the part of $y = f(x)$ for $x \geq 0$, and **reflect it in the y-axis**; the part for $x < 0$ disappears. $f(|x|)$ is always even.

Example 3

$f(x) = x^2 + 2x - 8$. Describe the graphs of $y = |f(x)|$ and $y = f(|x|)$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Zeros and vertex of f	$(x+4)(x-2)$: zeros $-4, 2$; vertex $(-1, -9)$
Step 2 $\|f\|$: reflect the part below the axis	between -4 and 2 the dip turns up: local maximum $(-1, 9)$
Step 3 $f(\|x\|)$: keep $x \geq 0$	from $(0, -8)$ up through $(2, 0)$
Step 4 Reflect it in the y-axis	zeros ± 2, minimum $(0, -8)$

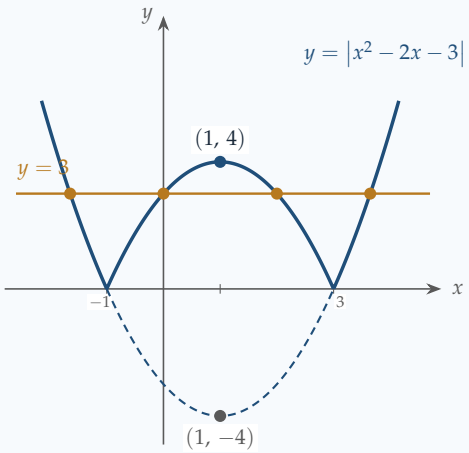


Figure 2 — Exam-style question 2: $f(x) = x^2 - 2x - 3$ (dashed where negative). $y = |f(x)|$ turns the dip between -1 and 3 up, so the vertex $(1, -4)$ becomes a local maximum $(1, 4)$; the line $y = 3$ then meets the graph four times.

APPLY

1 $y = |2x - 3|$: vertex and y-intercept.

2 $f(x) = (x-3)^2$: the zeros of $f(|x|)$.

II.2 Equations and inequalities with a modulus

Property 2 *equations with a modulus*

$|f(x)| = k$, $k > 0$, splits into $f(x) = k$ or $f(x) = -k$. For $|f(x)| = g(x)$, solve $f(x) = g(x)$ and $f(x) = -g(x)$, then **check each solution**: $g(x)$ must be ≥ 0. For $|f(x)| = |g(x)|$, square both sides, or solve $f = \pm g$.

Example 4

Solve $|x - 4| = 2x + 1$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 $f = g$	$x - 4 = 2x + 1$: $x = -5$
Step 2 Check $g \geq 0$	$2(-5) + 1 = -9 < 0$: rejected
Step 3 $f = -g$	$x - 4 = -(2x + 1)$: $x = 1$
Step 4 Check $g \geq 0$	$2(1) + 1 = 3 \geq 0$: $x = 1$

Watch out 2

$|x - 1| < 3$ means $-3 < x - 1 < 3$, one interval; $|x - 1| > 3$ means $x - 1 < -3$ or $x - 1 > 3$, two. A solution of $f(x) = -g(x)$ is false if $g(x) < 0$ there.

APPLY

3 $|x + 3| < 5$.

4 $|2x - 1| \geq 7$.

5 $|x^2 - 5| = 4$.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A Polynomial and rational inequalities

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 6]

Answer the following without a calculator.

(a) Solve the inequality $\dfrac{2x+1}{x-3} \leq 1$. [4]

(b) Solve the inequality $x^3 > 4x$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 One side zero, one fraction	$\dfrac{2x+1}{x-3} - 1 \leq 0, \ \dfrac{x+4}{x-3} \leq 0$ *M1 for one side zero, not multiplying by $x-3$.*	M1A1
Step 2 Critical values	critical values -4 and 3	(A1)
Step 3 Signs, 3 excluded	$-4 \leq x < 3$ *Both inequality signs right.*	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 One side zero, factorise	$x(x-2)(x+2) > 0$	M1
Step 2 Signs	$-2 < x < 0$ or $x > 2$	A1

B Modulus graphs and equations

EXAM-STYLE QUESTION 2 Paper 1 · no calculator · [Maximum mark: 7]

Let $f(x) = x^2 - 2x - 3$.

(a) Find the x-intercepts of the graph of $y = f(x)$, and the coordinates of its vertex. [2]

(b) Hence write down the coordinates of the local maximum point of the graph of $y = |f(x)|$. [1]

(c) Solve the equation $|f(x)| = 3$. [4]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Zeros	$x = -1,\ x = 3$	A1
Step 2 Vertex	vertex $(1, -4)$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Reflect the vertex	$(1, 4)$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Split: $f = 3$ or $f = -3$	$f(x) = 3$ or $f(x) = -3$	M1
Step 2 Solve $f = 3$	$x^2 - 2x - 6 = 0$: $x = 1 \pm \sqrt{7}$	A1
Step 3 Solve $f = -3$	$x^2 - 2x = 0$: $x = 0,\ x = 2$	A1
Step 4 All four	four solutions: $1 - \sqrt{7}, 0, 2, 1 + \sqrt{7}$	A1

C On the GDC (Paper 2)

EXAM-STYLE QUESTION 3 Paper 2 · calculator · [Maximum mark: 6]

Consider the graphs of $y = |x^2 - 4|$ and $y = 2x + 1$.

(a) Find the coordinates of the points of intersection of the two graphs. [2]

(b) Hence solve the inequality $|x^2 - 4| \geq 2x + 1$. [2]

(c) Find the exact value of the larger x-coordinate in part (a). [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Intersections on the GDC	$(1,\ 3)$ and $(3.45,\ 7.90)$	A1A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Read the intervals	$x \leq 1$ or $x \geq 3.45$	A1A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The equation of the right branch	$x^2 - 4 = 2x + 1,\ x^2 - 2x - 5 = 0$	M1
	For $x > 2$ the modulus is $x^2 - 4$ itself.	
Step 2 Exact root	$x = 1 + \sqrt{6}$	A1

IV WHICH METHOD? READ THE QUESTION

AHL 2.15, 2.16 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"solve $\frac{p(x)}{q(x)} \leq c$*"*	one side zero, one fraction	critical values, then signs P1
a denominator in an inequality	never multiply by it	its zero is always excluded W1
$y = \lvert f(x) \rvert$	reflect the parts below	in the x-axis D1
$y = f(\lvert x \rvert)$	keep $x \geq 0$	reflect it in the y-axis D1
$\lvert f(x) \rvert = g(x)$	$f = g$ or $f = -g$	check $g(x) \geq 0$ for each P2
$\lvert f(x) \rvert \geq g(x)$ *on Paper 2*	graph both	intersections, then read the intervals P2

In the exam 1 *command terms*

Solve the inequality: intervals, with $<$ or $\leq$ exactly right. **Hence write down**: read from the graph just found, no working. **Solve** $\lvert f(x) \rvert = k$: every solution, from both cases.

V ON THE GDC: PAPER 2

AHL 2.15, 2.16 · NumWorks · every calculator move of this topic

1 A modulus inequality on the GDC

Grapher → Add an expression → abs(x^2-4) and 2x+1 → Calculate → Intersection

abs is in the Toolbox. Two intersections; read off where y_1 is above y_2. **On paper:** the intersections, then the intervals, with the inequality signs.

GRAPHER

```
y1=abs(x^2-4)
y2=2x+1
Intersection
x=1 y=3
x=3.44949 y=7.89898
```

ANSWERS TO THE APPLY QUESTIONS

I.1 $-2 < x < 4$

I.2 $x > -3,\ x \neq 1$

I.3 $\frac{1-x}{x+2} > 0$: $-2 < x < 1$

I.4 $x \leq 0$ or $x > 4$

II.1 $(\frac{3}{2}, 0);\ (0, 3)$

II.2 $x = \pm 3$

II.3 $-8 < x < 2$

II.4 $x \leq -3$ or $x \geq 4$

II.5 $x = \pm 1,\ x = \pm 3$

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