Odd, even and self-inverse functions

Symmetry saves work: an even graph is its own mirror image in the y-axis, an odd graph is unchanged by a half turn about the origin, and the graph of a self-inverse function is its own reflection in $y = x$. Each symmetry is an algebraic identity, checked by replacing x by $-x$, or x by $f(x)$.

I ODD AND EVEN FUNCTIONS

AHL 2.14 · $f(-x) = f(x)$, $f(-x) = -f(x)$

I.1 Odd and even

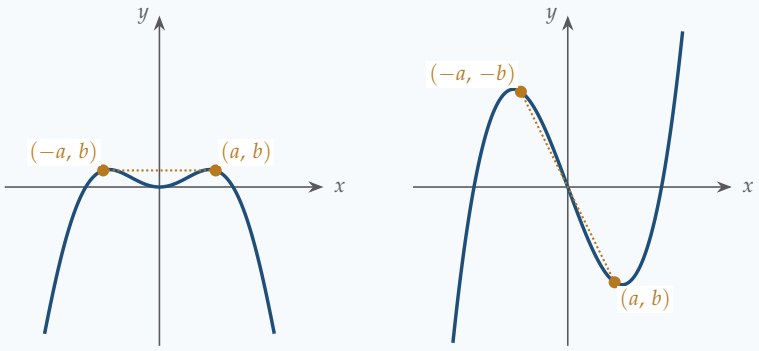
Definition 1 *odd and even functions*

f is **even** if $f(-x) = f(x)$ for every x in its domain: the graph is symmetric in the y-axis. f is **odd** if $f(-x) = -f(x)$: the graph has rotational symmetry of order 2 about the origin. The domain must itself be symmetric about 0.

Example 1

Show that $f(x) = \sin 2x + x$ is an odd function.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Replace x by $-x$	$f(-x) = \sin(-2x) + (-x)$
Step 2 Use $\sin(-\theta) = -\sin\theta$	$= -\sin 2x - x$
Step 3 Compare with $f(x)$	$= -(\sin 2x + x) = -f(x)$: odd



even: $y = x^2 \cos x$ odd: $y = x^3 - 4x$

Figure 1 — Exam-style question 1: $y = x^3 - 4x$ is odd, (a, b) and $(-a, -b)$ are both on it. $y = x^2 \cos x$ is even: (a, b) and $(-a, b)$ are both on it.

Watch out 1

To prove that a function is odd or even, compute $f(-x)$ **in general**; one value is not a proof. To prove it is neither, one counterexample is enough: $f(-1) \neq f(1)$ and $f(-1) \neq -f(1)$.

APPLY

1 $x^6 + \cos 3x$: odd, even or neither?

2 $4x^3 - x$?

3 $x + 1$?

I.2 Products and sums

Property 1 *products and sums*

Like signs: even $\times$ even and odd $\times$ odd are **even**; even $\times$ odd is **odd**. A sum of odd functions is odd, of even functions even. Most functions are neither: $x^2 + x$ is neither, since $(-x)^2 + (-x) \neq \pm(x^2 + x)$.

Example 2

Is $f(x) = x^3 \sin x$ odd, even or neither?

WHAT YOU DO	WHAT YOU WRITE
Step 1 Classify each factor	x^3 odd, $\sin x$ odd
Step 2 Like signs: even	odd $\times$ odd: f is even
Step 3 Check with $f(-x)$	$(-x)^3 \sin(-x) = (-x^3)(-\sin x) = x^3 \sin x$

APPLY

4 $x^2 \sin x$?

5 $(x^2 + 1)(x^3 - x)$?

II RESTRICTED DOMAINS AND SELF-INVERSE FUNCTIONS

AHL 2.14 · one-to-one, f^{-1}, $f \circ f = \mathrm{id}$, symmetry in $y = x$

II.1 Restricting the domain

Property 2 *restricting the domain*

Only a **one-to-one** function has an inverse. A function that is not one-to-one, such as a quadratic, gets an inverse on a restricted domain, usually from its vertex onwards: $f(x) = (x-2)^2 - 3$, $x \geq 2$, has $f^{-1}(x) = 2 + \sqrt{x+3}$, $x \geq -3$. The domain of f^{-1} is the range of f.

Example 3

$f(x) = x^2 + 2x - 3$, $x \geq k$. Find the smallest k for which f has an inverse, then $f^{-1}(x)$ and its domain.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Complete the square	$f(x) = (x+1)^2 - 4$
Step 2 Smallest k: the vertex	$k = -1$
Step 3 Swap and solve, the $+$ root	$x = (y+1)^2 - 4$: $y = -1 + \sqrt{x+4}$
The $+$ root, because $f^{-1}(x) \geq -1$.	
Step 4 Domain: the range of f	$f^{-1}(x) = -1 + \sqrt{x+4}$, $x \geq -4$

APPLY

1 $f(x) = (x+2)^2 - 5$, $x \leq -2$: $f^{-1}(x)$ and its domain.

II.2 Self-inverse functions

Definition 2 *self-inverse function*

f is **self-inverse** if $f^{-1} = f$, that is $f(f(x)) = x$ for every x in its domain. Its graph is symmetric in the line $y = x$. Examples: $\frac{1}{x}$, $c - x$, and $\frac{ax+b}{cx-a}$.

Example 4

Show that $f(x) = \dfrac{x+8}{x-1}$, $x \neq 1$, is self-inverse.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Substitute f into itself	$f(f(x)) = \dfrac{\frac{x+8}{x-1} + 8}{\frac{x+8}{x-1} - 1}$
Step 2 Multiply above and below by $x-1$	$= \dfrac{x+8+8x-8}{x+8-x+1}$
Step 3 Simplify	$= \dfrac{9x}{9} = x$: $f^{-1} = f$

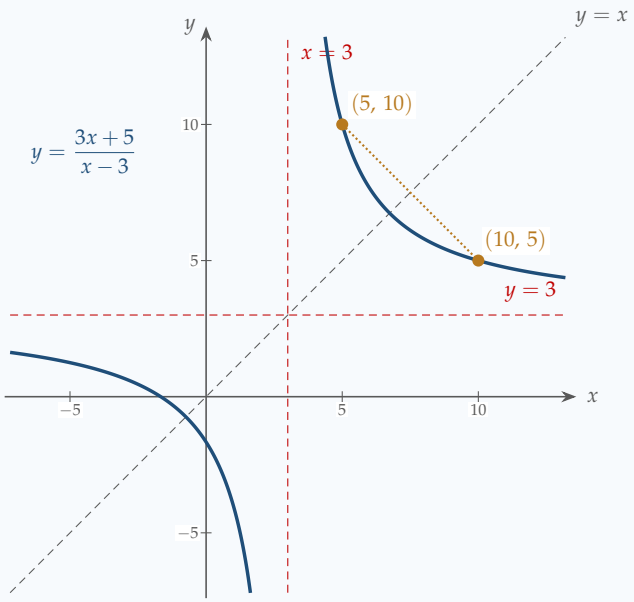


Figure 2 — Exam-style question 2: $y = \frac{3x+5}{x-3}$ is self-inverse. Reflecting any point (a, b) of the graph in $y = x$ gives (b, a), also on the graph; the asymptotes $x = 3$ and $y = 3$ are reflections of each other.

Watch out 2

$f^{-1}(x)$ is not $\frac{1}{f(x)}$. And if f is increasing, the graphs of f and f^{-1} meet only on $y = x$: solve $f(x) = x$ and keep the solutions in the domain. A decreasing f can meet its inverse elsewhere too.

APPLY

2 Is $f(x) = 6 - x$ self-inverse?

3 Where does $y = \dfrac{x+8}{x-1}$ meet $y = x$?

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A Odd and even functions

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 5]

Let $f(x) = x^3 - 4x$ and $g(x) = x^2 + \cos x$, for $x \in \mathbb{R}$.

(a) Show that f is an odd function. [2]

(b) Show that g is an even function. [1]

(c) Hence determine whether the function $h(x) = f(x)\,g(x)$ is odd, even or neither, justifying your answer. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Replace x by $-x$	$f(-x) = (-x)^3 - 4(-x) = -x^3 + 4x$	M1
Step 2 Compare with $f(x)$	$= -\left(x^3 - 4x\right) = -f(x)$	A1
	AG.	

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Replace x by $-x$, compare	$g(-x) = (-x)^2 + \cos(-x) = x^2 + \cos x = g(x)$	A1
	$\cos(-x) = \cos x$ *seen.*	

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Odd times even	$h(-x) = f(-x)g(-x) = -f(x)g(x) = -h(x)$	M1
Step 2 Conclude	h is odd	A1

B Self-inverse functions

EXAM-STYLE QUESTION 2 Paper 1 · no calculator · [Maximum mark: 7]

Let $f(x) = \dfrac{3x+5}{x-3}$, for $x \neq 3$.

(a) Show that $(f \circ f)(x) = x$. [3]

(b) Hence write down $f^{-1}(x)$. [1]

(c) Write down the equation of the line in which the graph of f is symmetric. [1]

(d) Find the x-coordinates of the points where the graph of f meets the line $y = x$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Substitute f into itself	$f(f(x)) = \dfrac{3\frac{3x+5}{x-3}+5}{\frac{3x+5}{x-3}-3}$	M1
Step 2 Multiply out	$= \dfrac{9x+15+5x-15}{3x+5-3x+9}$	A1
	Multiplying above and below by $x-3$.	
Step 3 Simplify	$= \dfrac{14x}{14} = x$	A1
	AG.	

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Self-inverse	$f^{-1}(x) = \dfrac{3x+5}{x-3}$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Line of symmetry	$y = x$	A1

WORKING (d)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Set $f(x) = x$	$3x+5 = x^2-3x,\ x^2-6x-5=0$	M1
Step 2 Solve	$x = 3 \pm \sqrt{14}$	A1

C A restricted domain, on the GDC (Paper 2)

EXAM-STYLE QUESTION 3 Paper 2 · calculator · [Maximum mark: 6]

Let $f(x) = x^2 - 4x + 1$, for $x \geq k$.

(a) Find the smallest value of k for which f has an inverse. [1]

(b) For this value of k, find $f^{-1}(x)$ and state its domain. [3]

(c) Solve the equation $f(x) = f^{-1}(x)$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Smallest k: the vertex	$k = 2$ *The vertex.*	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Swap and solve	$y = (x-2)^2 - 3,\ x = 2 + \sqrt{y+3}$ *Completing the square, or the quadratic formula; the + root because $x \geq 2$.*	M1
Step 2 The + root	$f^{-1}(x) = 2 + \sqrt{x+3}$	A1
Step 3 Domain: the range of f	$x \geq -3$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Solve $f(x) = x$	$x^2 - 4x + 1 = x$ *f is increasing, so f and f^{-1} meet on $y = x$.*	(M1)
Step 2 Keep the root in the domain	$x = 4.79$ *Exact value $\frac{5+\sqrt{21}}{2}$; the other root, 0.209, is not in the domain.*	A1

IV WHICH METHOD? READ THE QUESTION

AHL 2.14 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"show that f is odd (even)"	$f(-x)$ in general	$= -f(x)$ (or $= f(x)$) D1
"odd, even or neither"	like signs: even	neither: one counterexample P1
"the largest domain on which f^{-1} exists"	one-to-one	from the vertex, or up to it P2
"show that $f \circ f(x) = x$"	self-inverse	simplify the compound fraction D2
"symmetric in the line…"	self-inverse	$y = x$ D2
$f(x) = f^{-1}(x)$	f increasing	solve $f(x) = x$ in the domain W2

In the exam 1 *command terms*

Show that f is odd: $f(-x)$ computed for a general x. **Determine whether … justifying**: a conclusion and its reason. **Hence write down f^{-1}**: from the identity just shown, no working needed.

V ON THE GDC: PAPER 2

AHL 2.14 · NumWorks · every calculator move of this topic

1 A function and its inverse on the GDC

Grapher → Add an expression → f(x)=x^2-4x+1 with x≥2 → Calculate → Intersection

Draw f on its restricted domain and the line $y = x$; their intersection is where f meets f^{-1}, since f is increasing there. **On paper:** the equation $f(x) = x$, then x to three significant figures.

GRAPHER

```
y1=x^2-4x+1, x>=2
y2=x
Intersection
x=4.791288 y=4.791288
```

ANSWERS TO THE APPLY QUESTIONS

I.1 even

I.2 odd

I.3 neither: $f(1) = 2, f(-1) = 0$

I.4 even × odd: odd

I.5 odd

II.1 $-2 - \sqrt{x+5},\ x \geq -5$

II.2 yes: $6 - (6 - x) = x$

II.3 $x^2 - 2x - 8 = 0$: $x = 4, x = -2$

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