Partial fractions

Adding $\frac{-2}{x+1} + \frac{5}{2x-1}$ gives one fraction; partial fractions go the other way. The point is integration: $\frac{x+7}{(x+1)(2x-1)}$ has no antiderivative you can read off, but each simple fraction integrates to a logarithm. In the IB course the denominator has at most two distinct linear factors, and the numerator has a lower degree.

I SPLITTING A FRACTION

AHL 1.11 · $\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$

I.1 The partial fraction form

Definition 1 *partial fractions*

When the numerator has a lower degree than the denominator and the denominator has two distinct linear factors,

$$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$$

for constants A and B. This is the **partial fraction** form.

Example 1

Write $\dfrac{3}{x-2} + \dfrac{1}{x+4}$ as a single fraction.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Common denominator	$\dfrac{3(x+4)+(x-2)}{(x-2)(x+4)}$
Step 2 Add the numerators	$\dfrac{4x+10}{(x-2)(x+4)}$
Partial fractions go the other way: from $\frac{4x+10}{(x-2)(x+4)}$ back to the two parts.	

APPLY

1 $\dfrac{2}{x+1} - \dfrac{1}{x+5}$ as one fraction.

2 The form, with A and B, for $\dfrac{2x+9}{(x+3)(2x-1)}$.

I.2 Finding A and B

Property 1 *finding A and B*

Multiply both sides by the denominator: $px+q = A(cx+d)+B(ax+b)$, true for **every** x. Put $x = -\frac{b}{a}$ to remove B, then $x = -\frac{d}{c}$ to remove A; or equate the coefficients of x and the constants. For $\frac{x+7}{(x+1)(2x-1)}$: $x+7 = A(2x-1)+B(x+1)$; $x=-1$ gives $A=-2$, $x=\frac{1}{2}$ gives $B=5$.

Example 2

Express $\dfrac{7x-8}{(x-2)(x+1)}$ in partial fractions.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Multiply by the denominator	$7x-8 = A(x+1)+B(x-2)$
Step 2 $x=2$ removes B	$6=3A$: $A=2$
Step 3 $x=-1$ removes A	$-15=-3B$: $B=5$
Step 4 Write, then test $x=0$ At $x=0$: $\frac{-8}{-2}=4$ and $\frac{2}{-2}+\frac{5}{1}=4$.	$\dfrac{2}{x-2}+\dfrac{5}{x+1}$

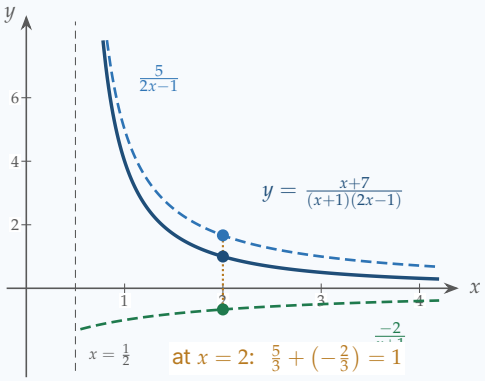


Figure 1 — Exam-style question 1 (a), for $x>\frac{1}{2}$: the graph of $\frac{x+7}{(x+1)(2x-1)}$ is the sum of the graphs of $\frac{5}{2x-1}$ and $\frac{-2}{x+1}$, height by height.

Watch out 1

Check by adding the two fractions back, or by testing one value such as $x=0$: $\frac{7}{-1}=-7$ and $\frac{-2}{1}+\frac{5}{-1}=-7$. Not in the course: repeated factors, quadratic factors, a numerator of equal or higher degree.

APPLY

3 $\dfrac{2x-5}{(2x+1)(x-1)}$ in partial fractions.

4 $\dfrac{6}{(x-1)(x+5)}$ in partial fractions.

II INTEGRATING WITH PARTIAL FRACTIONS

AHL 1.11, 5.15 · $\int \frac{A}{ax+b}\,\mathrm{d}x = \frac{A}{a}\ln|ax+b|+c$

II.1 Each part is a logarithm

Property 2 *each part is a logarithm*

$$\int \frac{A}{ax+b}\,\mathrm{d}x = \frac{A}{a}\ln|ax+b|+c.$$

Then the laws of logarithms combine the terms: $p\ln u + q\ln v = \ln(u^p v^q)$, which is how the answers are asked, "in the form $\ln k$".

Example 3

Find $\int \frac{3}{2x+5}\,\mathrm{d}x$ and $\int \frac{7x-8}{(x-2)(x+1)}\,\mathrm{d}x$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Divide by the coefficient of x	$\int \frac{3}{2x+5}\,\mathrm{d}x = \frac{3}{2}\ln|2x+5|+c$
Step 2 Split first (Example 2)	$\frac{2}{x-2}+\frac{5}{x+1}$
Step 3 Integrate each part	$2\ln|x-2|+5\ln|x+1|+c$

Watch out 2

$\int \frac{5}{2x-1}\,\mathrm{d}x = \frac{5}{2}\ln|2x-1|$, not $5\ln|2x-1|$: divide by the coefficient of x. And $\int \frac{1}{3-x}\,\mathrm{d}x = -\ln|3-x|$.

APPLY

1 $\int \frac{4}{3-2x}\,\mathrm{d}x.$

2 $\int \frac{6}{(x-1)(x+5)}\,\mathrm{d}x.$

II.2 Exact values in the form $\ln k$

Example 4

Show that $\int_3^5 \frac{7x-8}{(x-2)(x+1)}\,\mathrm{d}x = \ln\frac{2187}{32}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Antiderivative (Example 3)	$\left[2\ln(x-2)+5\ln(x+1)\right]_3^5$
Step 2 Substitute the limits	$2\ln 3+5\ln 6-(2\ln 1+5\ln 4) = 2\ln 3+5\ln\frac{3}{2}$
Step 3 One logarithm	$\ln\left(9\times\frac{243}{32}\right) = \ln\frac{2187}{32}$

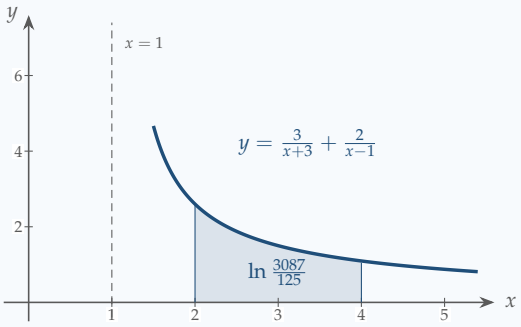


Figure 2 — Exam-style question 3 (b): the area under $y = \frac{5x+3}{(x+3)(x-1)} = \frac{3}{x+3} + \frac{2}{x-1}$ *from* $x = 2$ *to* $x = 4$ *is* $3\ln\frac{7}{5} + 2\ln 3 = \ln\frac{3087}{125} \approx 3.21$.

APPLY

3 $\displaystyle\int_2^3 \frac{6}{(x-1)(x+5)}\,\mathrm{d}x$ in the form $\ln k$.

II.3 An unknown limit

Property 3 *an unknown limit*

When the value of the integral is given and a limit k is unknown, the exact integral gives an equation in k: combine into one logarithm, then remove the ln; on Paper 2 the GDC solves it.

Example 5

Find $k > 2$ such that $\displaystyle\int_2^k \frac{6}{(x-1)(x+5)}\,\mathrm{d}x = \ln 2$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 The exact integral	$\ln(k-1) - \ln(k+5) + \ln 7$
Step 2 One logarithm	$\ln\dfrac{7(k-1)}{k+5} = \ln 2$
Step 3 Remove the ln	$7(k-1) = 2(k+5)$
Step 4 Solve	$k = \frac{17}{5}$

APPLY

4 Paper 2: $\displaystyle\int_3^k \frac{7x-8}{(x-2)(x+1)}\,\mathrm{d}x = 4$.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A Partial fractions, then an exact integral

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 7]

Let $f(x) = \dfrac{x+7}{(x+1)(2x-1)}$, for $x > \frac{1}{2}$.

(a) Express $f(x)$ in partial fractions. [3]

(b) Hence show that $\displaystyle\int_1^2 f(x)\,\mathrm{d}x = \ln\left(4\sqrt{3}\right)$. [4]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Multiply by the denominator	$x+7 = A(2x-1) + B(x+1)$	M1
Step 2 Substitute the roots	$x = -1$: $A = -2$; $x = \frac{1}{2}$: $B = 5$, so $f(x) = -\dfrac{2}{x+1} + \dfrac{5}{2x-1}$	A1A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Integrate each part	$\left[-2\ln(x+1) + \frac{5}{2}\ln(2x-1)\right]_1^2$	M1A1
	M1 for logarithms, A1 for the factor $\frac{5}{2}$.	
Step 2 Substitute the limits	$= -2\ln 3 + \frac{5}{2}\ln 3 + 2\ln 2 - 0$	A1
Step 3 One logarithm	$= \frac{1}{2}\ln 3 + 2\ln 2 = \ln\left(4\sqrt{3}\right)$	A1
	AG: the last step written.	

EXAM-STYLE QUESTION 2

Paper 1 · no calculator · [Maximum mark: 8]

Let $f(x) = \dfrac{4}{(x+1)(3-x)}$, for $-1 < x < 3$.

(a) Express $f(x)$ in partial fractions. [2]

(b) Hence show that $\displaystyle\int_0^k f(x)\,\mathrm{d}x = \ln\left(\frac{3(k+1)}{3-k}\right)$, where $0 < k < 3$. [4]

(c) Find the value of k for which $\displaystyle\int_0^k f(x)\,\mathrm{d}x = \ln 6$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Substitute the roots	$4 = A(3-x) + B(x+1)$: $A = 1,\ B = 1$	A1A1
	$f(x) = \frac{1}{x+1} + \frac{1}{3-x}$	

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Integrate each part	$[\ln(x+1) - \ln(3-x)]_0^k$	M1A1
	A1 for the minus sign of $-\ln(3-x)$.	
Step 2 Substitute the limits	$= \ln(k+1) - \ln(3-k) - (0 - \ln 3)$	A1
Step 3 One logarithm	$= \ln\left(\frac{3(k+1)}{3-k}\right)$	A1
	AG.	

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Remove the logarithm	$\frac{3(k+1)}{3-k} = 6,\ 3k+3 = 18-6k$	M1
Step 2 Solve	$k = \frac{5}{3}$	A1

B On the GDC (Paper 2)

EXAM-STYLE QUESTION 3 Paper 2 · calculator · [Maximum mark: 7]

Let $f(x) = \dfrac{5x+3}{(x+3)(x-1)}$, for $x > 1$.

(a) Express $f(x)$ in partial fractions. [2]

(b) Find the exact area of the region enclosed by the graph of f, the x-axis and the lines $x = 2$ and $x = 4$, giving your answer in the form $\ln \frac{p}{q}$, where $p, q \in \mathbb{Z}^+$. [3]

(c) The area enclosed by the graph of f, the x-axis and the lines $x = 2$ and $x = k$ is 2. Find the value of k. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Partial fractions	$f(x) = \dfrac{3}{x+3} + \dfrac{2}{x-1}$	A1A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Integrate, substitute the limits	$[3\ln(x+3) + 2\ln(x-1)]_2^4 = 3\ln 7 + 2\ln 3 - 3\ln 5$	M1A1
Step 2 One logarithm	$= \ln \frac{3087}{125}$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The equation in k	$3\ln \frac{k+3}{5} + 2\ln(k-1) = 2$	(M1)
Step 2 Solve on the GDC	$k = 3.04$	A1

IV WHICH METHOD? READ THE QUESTION

AHL 1.11 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"express in partial fractions"	$\frac{A}{ax+b}+\frac{B}{cx+d}$	multiply out, substitute the two roots P1
a check	recombine, or test $x=0$	both sides must agree W1
"hence find $\int$"	each part: $\frac{A}{a}\ln\lvert ax+b\rvert$	divide by the coefficient of x P2
"in the form $\ln k$"	laws of logarithms	$p\ln u+q\ln v=\ln u^p v^q$ P2
the integral is given, a limit unknown	one logarithm, then exponentiate	or the GDC on Paper 2 P3
$\frac{1}{3-x}$	coefficient -1	$-\ln\lvert 3-x\rvert$ W2

In the exam 1 *command terms*

Express in partial fractions: the values of A and B, then the fractions written out. **Hence show that**: the given answer reached, every log law visible (AG). **Exact**: logarithms and surds, no decimal.

V ON THE GDC: PAPER 2

AHL 1.11 · NumWorks · every calculator move of this topic

1 An integral and an unknown limit on the GDC

Calculation → Toolbox → Integral → int((5x+3)/((x+3)(x-1)),x,2,4)

The integral gives the decimal check of an exact answer. For an unknown limit, Equations: 3ln((k+3)/5)+2ln(k-1)=2, solved for k. **On paper:** the equation set up, then k to three significant figures.

CALCULATION

```
int((5x+3)/((x+3)(x-1)),2,4)
3.206641287
ln(3087/125)
3.206641287
```

ANSWERS TO THE APPLY QUESTIONS

I.1 $\dfrac{x+9}{(x+1)(x+5)}$

I.2 $\dfrac{A}{x+3}+\dfrac{B}{2x-1}$

I.3 $\dfrac{4}{2x+1}-\dfrac{1}{x-1}$

I.4 $\dfrac{1}{x-1}-\dfrac{1}{x+5}$

II.1 $-2\ln\lvert 3-2x\rvert+c$

II.2 $\ln\lvert x-1\rvert-\ln\lvert x+5\rvert+c$

II.3 $\ln 2-\ln 8+\ln 7=\ln\frac{7}{4}$

II.4 $2\ln(k-2)+5\ln\frac{k+1}{4}=4$: $k=4.85$

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