Polynomials and their roots

A polynomial of degree n has at most n real roots, and exactly n complex ones counted with repetition. Two facts do most of the work: the value $P(a)$ is the remainder when $P(x)$ is divided by $x - a$, and the coefficients alone give the sum and the product of the roots — without finding a single root.

I THE FACTOR AND REMAINDER THEOREMS

AHL 2.12 · division, $P(a)$, factor $x - a$, real coefficients

I.1 Polynomial division

Definition 1 *polynomial division*

Dividing $P(x)$ by $x - a$ gives a **quotient** $Q(x)$, of degree one less, and a **remainder** R, a constant:

$$P(x) = (x - a)\,Q(x) + R.$$

Dividing by a quadratic leaves a remainder of degree at most 1.

Example 1

Divide $x^3 + 2x^2 - 5x + 1$ by $x - 2$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 The form of the answer	$x^3 + 2x^2 - 5x + 1 = (x - 2)(x^2 + px + q) + R$
Step 2 Divide, term by term Long division, or the coefficients $1, 2, -5, 1$ with 2: $1,\ 4,\ 3$, remainder 7.	$Q(x) = x^2 + 4x + 3$
Step 3 The remainder	$R = 7$
Step 4 Write the identity	$x^3 + 2x^2 - 5x + 1 = (x - 2)(x^2 + 4x + 3) + 7$

APPLY

1 Divide $2x^3 - x + 4$ by $x + 1$.

I.2 The remainder theorem

Property 1 *the remainder theorem*

Putting $x = a$ in $P(x) = (x-a)Q(x) + R$ gives $R = P(a)$: **the remainder on division by $x - a$ is $P(a)$.** On division by $ax - b$ it is $P\left(\frac{b}{a}\right)$.

Example 2

Find the remainder when $3x^3 - 4x + 7$ is divided by $x + 2$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 The value of x that makes the divisor 0	$x + 2 = 0$: $x = -2$
Step 2 Substitute	$P(-2) = -24 + 8 + 7$
Step 3 The remainder	$R = -9$

APPLY

2 $x^4 + 2x - 5$ divided by $x - 1$: the remainder.

3 $4x^3 + x$ divided by $2x - 1$: the remainder.

I.3 The factor theorem

Property 2 *the factor theorem*

$x - a$ is a factor of $P(x)$ **if and only if** $P(a) = 0$. Two pieces of information (a factor, a remainder) give two equations for two unknown coefficients.

Example 3

Show that $x - 1$ is a factor of $P(x) = x^3 - 2x^2 - 5x + 6$, then factorise $P(x)$ fully.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Factor: $P(1) = 0$	$P(1) = 1 - 2 - 5 + 6 = 0$
Step 2 Divide by the factor	$P(x) = (x-1)(x^2 - x - 6)$
Step 3 Factorise the quadratic	$P(x) = (x-1)(x+2)(x-3)$

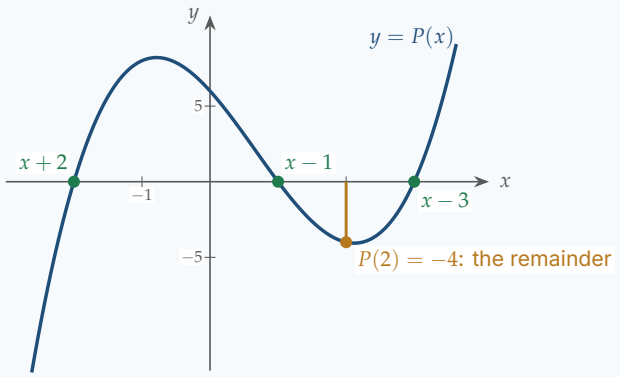


Figure 1 — Example 3: the remainder on division by $x - a$ is the height $P(a)$. At $a = -2$, 1 and 3 the remainder is 0: these are the factors.

Watch out 1

> The remainder on division by $x+1$ is $P(-1)$, not $P(1)$. And a cubic with real coefficients always has a real root, but its other two roots may be a conjugate pair.

APPLY

4 $x+3$ is a factor of x^3+kx^2-2x+3: find k.

5 x^3+ax+b has the factor $x-1$, and remainder 9 on division by $x-2$.

II SUMS AND PRODUCTS OF ROOTS

AHL 2.12 · $\sum\alpha = -\frac{a_{n-1}}{a_n}$, $\prod\alpha = \frac{(-1)^n a_0}{a_n}$

II.1 Sum and product of the roots

Property 3 *sum and product of the roots*

For $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$, with all n roots counted (complex and repeated ones included):

$$\text{sum of the roots} = -\frac{a_{n-1}}{a_n}, \qquad \text{product of the roots} = \frac{(-1)^n a_0}{a_n}.$$

Both are in the formula booklet.

Example 4

Without solving, find the sum and the product of the roots of $x^3 - 6x^2 + 5x + 12 = 0$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Read n, a_n, a_{n-1}, a_0	$n = 3,\ a_3 = 1,\ a_2 = -6,\ a_0 = 12$
Step 2 Sum	$-\dfrac{-6}{1} = 6$
Step 3 Product, with $(-1)^n$	$\dfrac{(-1)^3 \times 12}{1} = -12$
The roots are $-1, 3, 4$: their sum is 6, their product -12.	

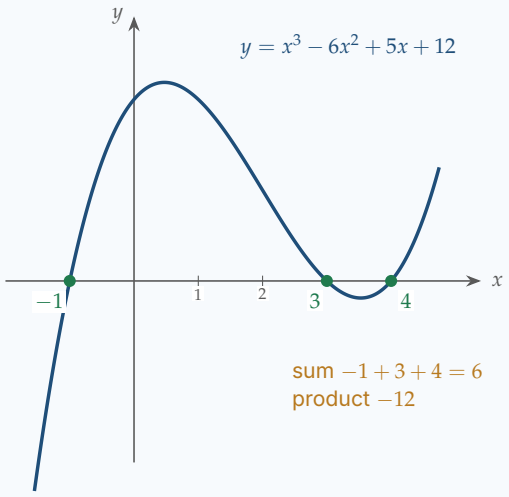


Figure 2 — Example 4: roots -1, 3 and 4, with sum $6 = -\frac{-6}{1}$ and product $-12 = \frac{(-1)^3 \times 12}{1}$.

Watch out 2

The sign of the product alternates with the degree: $(-1)^n$. And a degree-4 equation has four roots in the formula, even if only two of them are real.

APPLY

1 $2x^4 - 3x^3 + x - 8 = 0$: sum and product of the roots.

2 $x^3 + 5x - 2 = 0$: sum and product.

II.2 Products in pairs

Property 4 *why it works*

For a cubic,

$$a(x-\alpha)(x-\beta)(x-\gamma) = a\left(x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma\right):$$

compare the coefficients. The middle one gives the sum of the products in pairs, $\frac{a_1}{a_3}$, useful for $\alpha^2+\beta^2+\gamma^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$.

Example 5

The roots of $x^3 - 4x^2 + x + 6 = 0$ are α, β, γ. Find $\alpha^2+\beta^2+\gamma^2$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Sum	$\sum\alpha = 4$
Step 2 Products in pairs	$\sum\alpha\beta = \frac{1}{1} = 1$
Step 3 Squares from sum and pairs	$4^2 - 2\times 1 = 14$

Check: the roots are -1, 2, 3, and $1+4+9 = 14$.

APPLY

3 $x^3 + 2x^2 - 3x - 1 = 0$: $\alpha^2+\beta^2+\gamma^2$.

4 The roots of $x^3 - 12x^2 + 39x - 28 = 0$ are $a-d$, a, $a+d$: find them.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A The factor and remainder theorems

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 7]

Let $P(x) = 2x^3 + ax^2 + bx - 6$, where $a, b \in \mathbb{R}$. The polynomial $x - 2$ is a factor of $P(x)$, and when $P(x)$ is divided by $x + 1$ the remainder is -6.

(a) Find the value of a and the value of b. [4]

(b) Hence show that the equation $P(x) = 0$ has exactly one real root. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Factor: $P(2) = 0$	$P(2) = 0$: $16 + 4a + 2b - 6 = 0$ *Using the factor theorem.*	M1
Step 2 Remainder: $P(-1) = -6$	$P(-1) = -6$: $-2 + a - b - 6 = -6$ *Using the remainder theorem.*	M1
Step 3 Two equations	$2a + b = -5$ and $a - b = 2$	A1
Step 4 Solve	$a = -1,\ b = -3$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Divide by the factor	$P(x) = (x - 2)\left(2x^2 + 3x + 3\right)$ *M1 for dividing by $x - 2$ or comparing coefficients.*	M1A1
Step 2 Discriminant of the quadratic	$\Delta = 9 - 24 = -15 < 0$: the quadratic has no real root, so $x = 2$ is the only real root	R1

B Sums and products of roots

EXAM-STYLE QUESTION 2 Paper 1 · no calculator · [Maximum mark: 6]

The roots of the equation $x^3 + px^2 + qx - 64 = 0$, where $p, q \in \mathbb{R}$, are α, 2α and 4α.

(a) Find the value of α. [3]

(b) Hence find the value of p and the value of q. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Product of the roots	product of the roots $= \alpha \times 2\alpha \times 4\alpha = 8\alpha^3$	A1
Step 2 The formula for the product	$8\alpha^3 = \frac{(-1)^3(-64)}{1} = 64$ *The product formula with the correct sign.*	M1
Step 3 Solve	$\alpha = 2$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Sum of the roots	sum $2 + 4 + 8 = 14 = -p$: $p = -14$	A1
Step 2 Products in pairs	$q = 2 \times 4 + 4 \times 8 + 2 \times 8$ *The sum of the products in pairs.*	(M1)
Step 3 Compute	$q = 56$	A1

C On the GDC (Paper 2)

EXAM-STYLE QUESTION 3 Paper 2 · calculator · [Maximum mark: 6]

Consider the equation $x^4 - 2x^3 - 7x^2 + 8x + 5 = 0$, with roots α, β, γ and δ.

(a) Find the four roots. [2]

(b) Write down the value of $\alpha\beta\gamma\delta$. [1]

(c) Find the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Roots on the GDC	$-2.25,\ -0.466,\ 1.47,\ 3.25$	A2
	A1 for two or three correct roots.	

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Product of the roots	$\alpha\beta\gamma\delta = \frac{(-1)^4 \times 5}{1} = 5$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Sum and pairs	$\sum\alpha = 2,\ \sum\alpha\beta = -7$	(A1)
Step 2 Squares from sum and pairs	$2^2 - 2 \times (-7)$	(M1)
Step 3 Compute	$= 18$	A1

IV WHICH METHOD? READ THE QUESTION

AHL 2.12 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"$x - a$ is a factor"	$P(a) = 0$	one equation in the unknown coefficients P2
"the remainder when divided by $x - a$ is R"	$P(a) = R$	by $x + 1$: $P(-1)$ P1
"hence factorise", "hence solve"	divide by the known factor	then the quadratic: its discriminant D1
"the sum of the roots"	$-\frac{a_{n-1}}{a_n}$	no need to find them P3
"the product of the roots"	$\frac{(-1)^n a_0}{a_n}$	watch the sign P3
$\alpha^2 + \beta^2 + \cdots$	$(\sum\alpha)^2 - 2\sum\alpha\beta$	pairs from the next coefficient P4

In the exam 1 *command terms*

Hence show that ... exactly one real root: the factor, then the discriminant of the quadratic and a sentence. **Find the value of**: an exact value, from the coefficients. **Write down**: a sum or a product read straight from the equation.

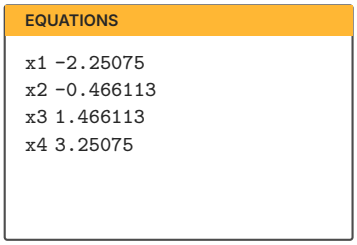
V ON THE GDC: PAPER 2

AHL 2.12 · NumWorks · every calculator move of this topic

1 Roots of a polynomial on the GDC

Equations → Add an equation → x^4-2x^3-7x^2+8x+5=0 → Solve

The NumWorks solves a polynomial equation of degree up to 3 exactly, and of any degree numerically. **On paper:** each root to three significant figures; a sum or product from the coefficients needs no calculator.



ANSWERS TO THE APPLY QUESTIONS

I.1 $Q = 2x^2 - 2x + 1,\ R = 3$

I.2 -2

I.3 $P\left(\frac{1}{2}\right) = 1$

I.4 $9k - 18 = 0$: $k = 2$

I.5 $a + b = -1, 2a + b = 1$: $a = 2, b = -3$

II.1 $\frac{3}{2};\ -4$

II.2 $0;\ 2$

II.3 $4 + 6 = 10$

II.4 $3a = 12, a = 4;\ 4(16 - d^2) = 28$: $1,\ 4,\ 7$

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