Polynomials and their roots

Skills sheet · HL AA · AHL 2.12

KEY FACTS

$P(x) = (x-a)Q(x) + R$ D1
Remainder on division by $x-a$: $P(a)$ P1
$x-a$ is a factor $\iff P(a) = 0$ P2
Sum $-\frac{a_{n-1}}{a_n}$, product $\frac{(-1)^n a_0}{a_n}$ P3
$\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$ P4

By $x+1$: $P(-1)$ W1.
Product: the sign $(-1)^n$ W2.
All n roots count, complex too W2.

WHAT UNLOCKS

- "is a factor", "the remainder is" $\rightarrow$ one equation each, then solve for the coefficients.
- "hence factorise" $\rightarrow$ divide by the known factor, then the quadratic.
- roots "in arithmetic (geometric) progression" $\rightarrow$ $a-d, a, a+d$ (or $\alpha, r\alpha, r^2\alpha$), then sum and product.
- one complex root, real coefficients $\rightarrow$ its conjugate too.

WORKED EXAMPLE ①

Find the remainder when $x^4 - 3x^2 + 5x - 2$ is divided by $x+2$. no GDC

$P(-2) = 16 - 12 - 10 - 2$ (M1)
$= -8$ A1

Method
A remainder
① $x+2=0$ gives $x=-2$.
② Substitute, no division.
③ The number is the remainder.

WORKED EXAMPLE ②

Find the sum and the product of the roots of $3x^4 - 5x^3 + x - 6 = 0$. no GDC

sum $= -\frac{-5}{3} = \frac{5}{3}$ A1
product $= \frac{(-1)^4 \times (-6)}{3} = -2$ A1

Method
Sum and product
① Read a_n, a_{n-1}, a_0.
② Missing terms: coefficient 0.
③ Sign of the product: $(-1)^n$.

FLUENCY

●○○ **1.** The remainder of $x^3 + 4x - 1$ divided by $x-1$ no GDC

●○○ **2.** Is $x+3$ a factor of $x^3 + 27$? no GDC

●○○ **3.** The remainder of $2x^3 - x + 1$ divided by $2x-1$ no GDC

●○○ **4.** The sum of the roots of $2x^3 + 6x^2 - x + 4 = 0$ no GDC

●○○ **5.** The product of the roots of $x^5 - 2x + 7 = 0$ no GDC

●○○ **6.** The sum of the roots of $x^4 - 10 = 0$ no GDC

REASONING

●●● **Question 1.** Let $P(x) = x^3 + kx^2 - 4x + 12$, where $k \in \mathbb{R}$. The polynomial $x + 2$ is a factor of $P(x)$. [no GDC]

(a) Find the value of k, and hence factorise $P(x)$ completely. [4]

●●● **Question 2.** When $P(x) = x^3 + ax + b$ is divided by $x - 1$ the remainder is 3, and when it is divided by $x + 2$ the remainder is -9. [no GDC]

(a) Find the value of a and the value of b. [4]

●●● **Question 3.** The roots of $2x^3 - 4x^2 + 3x - 5 = 0$ are α, β and γ. [no GDC]

(a) Find $\alpha + \beta + \gamma$, $\alpha\beta\gamma$ and $\alpha\beta + \beta\gamma + \gamma\alpha$, and hence $\alpha^2 + \beta^2 + \gamma^2$. [4]

PROBLEMS

●●● **Question 4.** The roots of $x^3 - 9x^2 + kx - 15 = 0$, where $k \in \mathbb{R}$, are in arithmetic progression. [no GDC]

(a) Find the three roots and the value of k. [6]

●●● **Question 5.** The polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ has real coefficients. Two of its roots are 2i and $1 - \mathrm{i}$. [no GDC]

(a) Write down the other two roots. [2]

(b) Find the values of a, b, c and d. [4]

ANSWERS

1 $P(1) = 4$

2 yes: $(-3)^3 + 27 = 0$

3 $P\left(\frac{1}{2}\right) = \frac{3}{4}$

4 -3

5 -7

6 0

Q1 $P(-2) = 4k + 12 = 0, k = -3;\ (x+2)(x^2 - 5x + 6) = (x+2)(x-2)(x-3)$

Q2 $a + b = 2,\ -2a + b = -1:\ a = 1, b = 1$

Q3 $2,\ \frac{5}{2},\ \frac{3}{2};\ 2^2 - 2 \times \frac{3}{2} = 1$

Q4 $a - d, a, a + d:\ 3a = 9, a = 3;\ 3(9 - d^2) = 15, d = \pm 2:$ roots $1, 3, 5;\ k = 3 + 15 + 5 = 23$

Q5 $-2\text{i},\ 1 + \text{i};\ (x^2 + 4)(x^2 - 2x + 2) = x^4 - 2x^3 + 6x^2 - 8x + 8:\ a = -2, b = 6, c = -8, d = 8$

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