Rational functions and oblique asymptotes

At SL the rational functions are $\frac{ax+b}{cx+d}$, with two straight asymptotes. At HL the degrees change: a linear numerator over a quadratic, whose graph returns to $y = 0$; or a quadratic over a linear denominator, whose graph follows a sloping line, an **oblique asymptote**. Division tells you which line, and the remainder tells you on which side the curve stays.

I QUADRATIC OVER LINEAR: OBLIQUE ASYMPTOTES

AHL 2.13 · $\frac{ax^2+bx+c}{dx+e} = mx + k + \frac{R}{dx+e}$

I.1 Dividing: the oblique asymptote

Definition 1 *oblique asymptote*

Dividing gives $\dfrac{ax^2+bx+c}{dx+e} = mx + k + \dfrac{R}{dx+e}$. As $x \to \pm\infty$ the last term tends to 0, so the graph approaches the line $y = mx + k$: an **oblique asymptote**. The vertical asymptote is $x = -\frac{e}{d}$ (if $R \neq 0$).

Property 1 *finding the line*

Divide the numerator by the denominator, or write $ax^2 + bx + c = (dx + e)(mx + k) + R$ and compare coefficients. For $\frac{x^2+x-2}{x-2}$: $x^2 + x - 2 = (x-2)(x+3) + 4$, so $y = x + 3 + \frac{4}{x-2}$: asymptotes $x = 2$ and $y = x + 3$.

Example 1

Find the asymptotes of the graph of $y = \dfrac{3x^2 - x + 2}{x + 1}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Write the identity	$3x^2 - x + 2 = (x+1)(mx+k) + R$
Step 2 Compare coefficients	x^2: $m = 3$; x: $m + k = -1$, $k = -4$; constant: $k + R = 2$, $R = 6$
Step 3 The divided form	$y = 3x - 4 + \dfrac{6}{x+1}$
Step 4 The asymptotes	$x = -1$ and $y = 3x - 4$

APPLY

1 $y = \dfrac{x^2 + 4x + 1}{x + 2}$: divided form and asymptotes.

2 $y = \dfrac{2x^2 - 3x}{x - 1}$: divided form.

I.2 Stationary points from the divided form

Property 2 *stationary points from the divided form*

The divided form is easier to differentiate: $y = x + 3 + 4(x-2)^{-1}$ gives $y' = 1 - \frac{4}{(x-2)^2}$, zero when $x - 2 = \pm 2$.

Example 2

Find the stationary points of $y = \dfrac{x^2 + x - 2}{x - 2}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 The divided form	$y = x + 3 + 4(x-2)^{-1}$
Step 2 Differentiate, $= 0$	$1 - \dfrac{4}{(x-2)^2} = 0$
Step 3 Solve	$(x-2)^2 = 4$: $x = 0$ or $x = 4$
Step 4 The points	$(0, 1)$ and $(4, 9)$

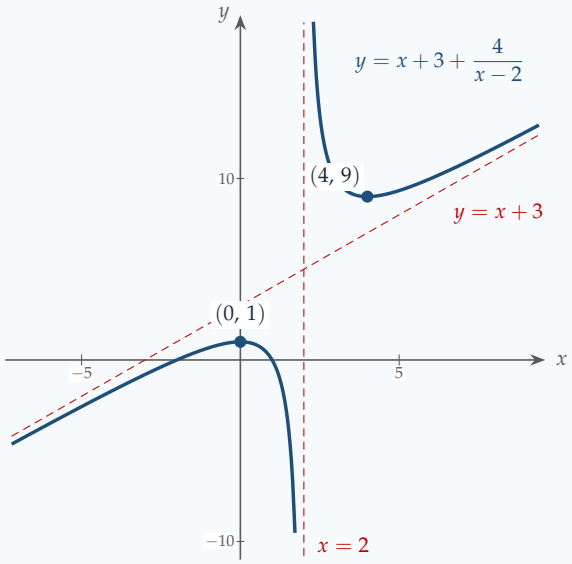


Figure 1 — Example 2: $y = x + 3 + \frac{4}{x-2}$ approaches $y = x + 3$, above it for $x > 2$ (the remainder term is positive) and below it for $x < 2$.

Watch out 1

The oblique asymptote is $y = mx + k$, the quotient — not the numerator divided by the leading term. And the curve never meets it when $R \neq 0$.

APPLY

3 Stationary points of $y = x + \dfrac{9}{x}$.

4 Stationary points of $y = 2x - 1 - \dfrac{1}{x-1}$.

II.1 Asymptotes and intercepts

Property 3 *the features*

For $y=\dfrac{ax+b}{cx^2+dx+e}$: the **horizontal asymptote** is $y=0$ (the denominator grows faster); the **vertical asymptotes** are at the real roots of the denominator — two, one or none, from its discriminant; the x-intercept is $x=-\frac{b}{a}$, the y-intercept $\frac{b}{e}$.

Example 3

Find the asymptotes and the intercepts of the graph of $y=\dfrac{2x-1}{x^2-x-2}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Factorise the denominator	$(x-2)(x+1)$
Step 2 Vertical asymptotes: its roots	$x=2$ and $x=-1$
Step 3 Horizontal asymptote	$y=0$
Step 4 Intercepts	$\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$

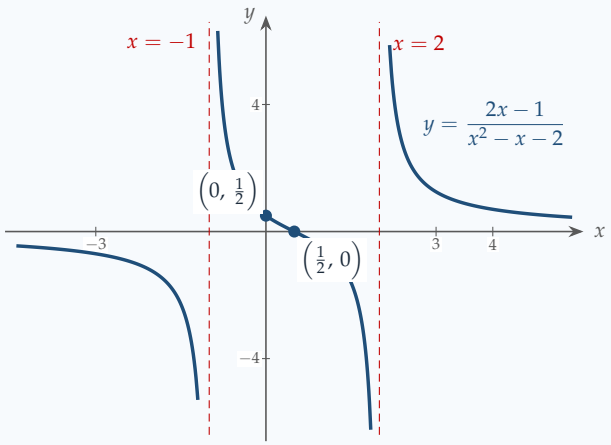


Figure 2 — Example 3: two vertical asymptotes, $x=-1$ and $x=2$, and $y=0$; the graph crosses its horizontal asymptote at $\left(\frac{1}{2},0\right)$.

APPLY

1 $y=\dfrac{x+3}{x^2+4x+4}$: asymptotes and intercepts.

2 $y=\dfrac{x-5}{x^2+1}$: vertical asymptotes?

II.2 No vertical asymptote: the range

Property 4 *no vertical asymptote: a range*

If the denominator has no real root, the graph is in one piece and bounded: its **range** runs from the minimum to the maximum value, found with the quotient rule. The graph can cross $y = 0$ (at the x-intercept); the asymptote describes $x \to \pm\infty$ only.

Example 4

Find the range of $g(x) = \dfrac{4x}{x^2+4}$, $x \in \mathbb{R}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 No vertical asymptote	$x^2 + 4 > 0$ for all x
Step 2 Quotient rule	$g'(x) = \dfrac{4(x^2+4) - 8x^2}{(x^2+4)^2} = \dfrac{16 - 4x^2}{(x^2+4)^2}$
Step 3 Stationary points	$x = \pm 2$: $g(2) = 1$, $g(-2) = -1$
Step 4 The range	$-1 \le g(x) \le 1$

Watch out 2

A curve may cross a horizontal asymptote; it cannot cross a vertical one. And a common factor of numerator and denominator gives a hole, not an asymptote.

APPLY

3 The range of $\dfrac{x+2}{x^2+5}$.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A Quadratic over linear

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 7]

Let $f(x) = \dfrac{x^2 - 3x + 6}{x - 1}$, for $x \neq 1$.

(a) Write $f(x)$ in the form $x + a + \dfrac{b}{x-1}$, where $a, b \in \mathbb{Z}$. [2]

(b) Write down the equations of the asymptotes of the graph of f. [2]

(c) Find the coordinates of the stationary points of the graph of f. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Write the identity	$x^2 - 3x + 6 = (x-1)(x-2) + 4$	(M1)
Step 2 The divided form	$f(x) = x - 2 + \dfrac{4}{x-1}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Vertical asymptote	$x = 1$	A1
Step 2 Oblique asymptote	$y = x - 2$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Differentiate the divided form, $= 0$	$f'(x) = 1 - \dfrac{4}{(x-1)^2} = 0$	M1
Step 2 Solve	$x - 1 = \pm 2$: $x = 3$ or $x = -1$	(A1)
Step 3 The points	$(3, 3)$ and $(-1, -5)$	A1

B Linear over quadratic

EXAM-STYLE QUESTION 2 Paper 1 · no calculator · [Maximum mark: 7]

Let $g(x) = \dfrac{x+1}{x^2+3}$, for $x \in \mathbb{R}$.

(a) Explain why the graph of g has no vertical asymptote, and write down the equation of its horizontal asymptote. [2]

(b) Find the x-coordinates of the stationary points of the graph of g. [3]

(c) Hence find the range of g. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 No real root in the denominator	$x^2 + 3 > 0$ for all x (no real root)	R1
Step 2 Horizontal asymptote	$y = 0$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Quotient rule	$g'(x) = \dfrac{(x^2+3) - 2x(x+1)}{(x^2+3)^2}$ *The quotient rule.*	M1
Step 2 Numerator $= 0$	$-x^2 - 2x + 3 = 0$	A1
Step 3 Solve	$x = 1,\ x = -3$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Values at the stationary points	$g(1) = \frac{1}{2},\ g(-3) = -\frac{1}{6}$	(A1)
Step 2 The range	$-\frac{1}{6} \le g(x) \le \frac{1}{2}$	A1

C On the GDC (Paper 2)

EXAM-STYLE QUESTION 3 Paper 2 · calculator · [Maximum mark: 7]

Let $h(x) = \dfrac{2x^2 + 3x - 4}{x + 2}$, for $x \neq -2$.

(a) Find the equations of the asymptotes of the graph of h. [3]

(b) Find the x-coordinates of the points where the graph of h meets the line $y = x + 3$. [2]

(c) Show that $h'(x) > 0$ for all $x \neq -2$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Write the identity	$2x^2 + 3x - 4 = (x + 2)(2x - 1) - 2$	(M1)
Step 2 Vertical asymptote	$x = -2$	A1
Step 3 Oblique asymptote	$y = 2x - 1$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Intersections on the GDC	-2.32 and 4.32	A1A1
	Exact values $1 \pm \sqrt{11}$.	

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Differentiate the divided form	$h(x) = 2x - 1 - \dfrac{2}{x+2}$, $h'(x) = 2 + \dfrac{2}{(x+2)^2}$	M1
Step 2 Both terms positive	both terms positive, so $h'(x) > 0$	R1
	AG.	

IV WHICH METHOD? READ THE QUESTION

AHL 2.13 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
degree 2 *over degree* 1	divide: $mx+k+\frac{R}{dx+e}$	oblique asymptote $y=mx+k$ D1
"in the form $x+a+\frac{b}{x-1}$*"*	compare coefficients	then a and b read off P1
stationary points	differentiate the divided form	$(x-c)^{-1}$, no quotient rule P2
degree 1 *over degree* 2	$y=0$	vertical asymptotes: the discriminant of the denominator P3
"range" of a bounded rational function	max and min by $g'(x)=0$	between them, both included P4
"the curve crosses its asymptote"	horizontal: possible	vertical: never W2

In the exam 1 *command terms*

Write down the equations of the asymptotes: two equations, $x=\ldots$ and $y=\ldots$. **Explain why**: a reason about the denominator. **Hence find the range**: from the stationary values just found, with the inequality signs right.

V ON THE GDC: PAPER 2

AHL 2.13 · NumWorks · every calculator move of this topic

1 A rational graph on the GDC

Grapher → Add an expression → (2x^2+3x-4)/(x+2) → Calculate → Intersection

Plot the curve and a line, then Calculate, Intersection for each point. The vertical asymptote shows as a gap; zoom out to see the oblique one. **On paper:** each coordinate to three significant figures.

GRAPHER

```
y1=(2x^2+3x-4)/(x+2)
y2=x+3
Intersection
x=-2.31662 y=0.683375
x=4.31662 y=7.31662
```

ANSWERS TO THE APPLY QUESTIONS

I.1 $x+2-\dfrac{3}{x+2}$; $x=-2$, $y=x+2$

I.2 $2x-1-\dfrac{1}{x-1}$

I.3 $(3,\,6)$, $(-3,\,-6)$

I.4 none: $y'=2+\dfrac{1}{(x-1)^2}>0$

II.1 $x=-2$, $y=0$; $(-3,\,0)$, $\left(0,\,\frac{3}{4}\right)$

II.2 none: $x^2+1>0$

II.3 stationary at $x=1$, $x=-5$: $-\frac{1}{10}\le y\le\frac{1}{2}$

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