# Rational functions and oblique asymptotes

Skills sheet · HL AA · AHL 2.13

## KEY FACTS

$\frac{ax^2+bx+c}{dx+e} = mx+k+\frac{R}{dx+e}$: $y = mx+k$ D1

Compare coefficients to find $m, k, R$ P1

Differentiate the divided form P2

$\frac{ax+b}{cx^2+dx+e}$: $y = 0$; vertical: roots below P3

No vertical asymptote: range from max and min P4

*The quotient is the asymptote* W1.

*A common factor: a hole* W2.

*$y = 0$ can be crossed* W2.

## WHAT UNLOCKS

- numerator one degree higher $\rightarrow$ divide: oblique asymptote and vertical asymptote.
- "stationary points" of $mx+k+\frac{R}{x-c}$ $\rightarrow$ $m-\frac{R}{(x-c)^2}=0$.
- "how many vertical asymptotes" $\rightarrow$ the discriminant of the denominator, and common factors.
- "range" $\rightarrow$ the stationary values, or the gaps left by the asymptotes.

## WORKED EXAMPLE ①

**Find the asymptotes of $y = \dfrac{2x^2+5x-1}{x+3}$.** no GDC

$2x^2+5x-1 = (x+3)(2x-1)+2$ (M1)

$x = -3,\ y = 2x-1$ A1A1

**Method**

*An oblique asymptote*

① Divide: quotient and remainder.
② Remainder $\neq 0$: vertical asymptote.
③ The quotient is the line.

## WORKED EXAMPLE ②

**Find the asymptotes and the intercepts of $y = \dfrac{x+4}{x^2-9}$.** no GDC

$x = 3,\ x = -3,\ y = 0$ A1A1

$(-4,\ 0),\ \left(0,\ -\frac{4}{9}\right)$ A1

**Method**

*Linear over quadratic*

① Roots of the denominator.
② Horizontal: $y = 0$.
③ Intercepts: $x = 0$, then $y = 0$.

## FLUENCY

●○○ **1.** The asymptotes of $y = \dfrac{x^2+1}{x}$ no GDC

●○○ **2.** The oblique asymptote of $y = \dfrac{x^2-4}{x-1}$ no GDC

●○○ **3.** The vertical asymptotes of $y = \dfrac{3x}{x^2+x-6}$ no GDC

●○○ **4.** How many vertical asymptotes for $y = \dfrac{x-2}{x^2+2x+5}$? no GDC

●○○ **5.** The horizontal asymptote of $y = \dfrac{5x+1}{2x^2+7}$ no GDC

●○○ **6.** The $y$-intercept of $y = \dfrac{x+6}{x^2-x-3}$ no GDC

## REASONING

●●● **Question 1.** Let $f(x) = \dfrac{2x^2 - x + 3}{x - 1}$, for $x \neq 1$. [no GDC]

(a) Find the equations of the asymptotes of the graph of $f$. [3]

●●● **Question 2.** Let $f(x) = x + 2 + \dfrac{9}{x - 1}$, for $x \neq 1$. [no GDC]

(a) Find the coordinates of the stationary points of the graph of $f$. [4]

●●● **Question 3.** Consider $y = \dfrac{x - 1}{x^2 - 4x + k}$, where $k \in \mathbb{R}$. [no GDC]

(a) Find the values of $k$ for which the graph has exactly one vertical asymptote. [4]

## PROBLEMS

●●● **Question 4.** Let $g(x) = \dfrac{2x}{x^2 + 1}$, for $x \in \mathbb{R}$. [no GDC]

(a) Find $g'(x)$, and hence the coordinates of the stationary points of the graph of $g$. [4]

(b) Write down the range of $g$. [1]

(c) Find the number of solutions of $g(x) = k$ when $0 < k < 1$. [2]

●●● **Question 5.** Let $f(x) = \dfrac{x^2 + kx + 4}{x + 1}$, for $x \neq -1$, where $k \in \mathbb{R}$. [no GDC]

(a) Show that $f(x) = x + k - 1 + \dfrac{5 - k}{x + 1}$. [2]

(b) Find the values of $k$ for which the graph of $f$ has no stationary point. [3]

## ANSWERS

**1** $x = 0,\ y = x$

**2** $x^2 - 4 = (x-1)(x+1) - 3$: $y = x + 1$

**3** $x = 2,\ x = -3$

**4** none: $\Delta = 4 - 20 < 0$

**5** $y = 0$

**6** $-2$

**Q1** $2x^2 - x + 3 = (x-1)(2x+1) + 4$: $x = 1,\ y = 2x + 1$

**Q2** $1 - \frac{9}{(x-1)^2} = 0$, $x = 4$ or $x = -2$: $(4,\ 9),\ (-2,\ -3)$

**Q3** $\Delta = 16 - 4k = 0$: $k = 4$ (asymptote $x = 2$); or $x = 1$ a root of the denominator: $k = 3$, $(x-1)(x-3)$, a hole at $x = 1$ and the asymptote $x = 3$

**Q4** $g'(x) = \frac{2-2x^2}{(x^2+1)^2}$: $(1,\ 1),\ (-1,\ -1)$; $-1 \le g(x) \le 1$; two solutions

**Q5** $(x+1)(x+k-1) + 5 - k$; $f'(x) = 1 - \frac{5-k}{(x+1)^2}$ never 0 when $5 - k \le 0$: $k \ge 5$

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