Complex numbers in Cartesian form

The equation $x^2 = -1$ has no real solution. Calling one solution i and computing with it like any other letter, with $\mathrm{i}^2 = -1$, gives numbers $a + b\mathrm{i}$ in which every quadratic equation has its roots. Most questions come down to one move: two complex numbers are equal when their real parts are equal **and** their imaginary parts are equal.

I THE CARTESIAN FORM AND ITS ALGEBRA

AHL 1.12 · $z = a + b\mathrm{i}$, Re, Im, z^*, equal parts

I.1 Real part, imaginary part, conjugate

Definition 1 *a complex number*

A **complex number** is $z = a + b\mathrm{i}$ with $a, b \in \mathbb{R}$ and $\mathrm{i}^2 = -1$. The **real part** is $\mathrm{Re}(z) = a$, the **imaginary part** is $\mathrm{Im}(z) = b$ (a real number, without the i). The **conjugate** is $z^* = a - b\mathrm{i}$.

Example 1

$z = -2 + 5\mathrm{i}$. Write down $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ and z^*, then find $3z - z^*$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Read a and b	$\mathrm{Re}(z) = -2,\ \mathrm{Im}(z) = 5$
Step 2 Conjugate: change the sign of b	$z^* = -2 - 5\mathrm{i}$
Step 3 Compute	$3z - z^* = -6 + 15\mathrm{i} + 2 + 5\mathrm{i} = -4 + 20\mathrm{i}$

Watch out 1

$\mathrm{Im}(3 - 4\mathrm{i}) = -4$, not $-4\mathrm{i}$. And z^* changes the sign of the imaginary part only: $(x + y\mathrm{i})^* = x - y\mathrm{i}$, so $\mathrm{i}z^* = \mathrm{i}x + y$.

APPLY

1 Re and Im of $(4 - 7\mathrm{i})^*$.

2 $\mathrm{i}z^*$ for $z = 3 + \mathrm{i}$.

I.2 Adding, multiplying, dividing

Property 1 *computing in Cartesian form*

Add and multiply as with brackets, then replace i^2 by -1:

$$(2+3\mathrm{i})(1-\mathrm{i}) = 2-2\mathrm{i}+3\mathrm{i}-3\mathrm{i}^2 = 5+\mathrm{i}.$$

Since $zz^* = a^2+b^2$ is real, divide by multiplying above and below by the conjugate of the denominator: $\dfrac{5+\mathrm{i}}{1-\mathrm{i}} = \dfrac{(5+\mathrm{i})(1+\mathrm{i})}{1^2+1^2} = 2+3\mathrm{i}$.

Example 2

Find $(3-2\mathrm{i})(4+\mathrm{i})$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Expand	$12+3\mathrm{i}-8\mathrm{i}-2\mathrm{i}^2$
Step 2 Replace i^2 by -1, collect	$14-5\mathrm{i}$

Example 3

Write $\dfrac{7+\mathrm{i}}{3-\mathrm{i}}$ in the form $a+b\mathrm{i}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Multiply by the conjugate of the denominator	$\dfrac{(7+\mathrm{i})(3+\mathrm{i})}{(3-\mathrm{i})(3+\mathrm{i})}$
Step 2 Expand: the denominator is real	$\dfrac{21+10\mathrm{i}+\mathrm{i}^2}{9+1} = \dfrac{20+10\mathrm{i}}{10}$
Step 3 Split	$2+\mathrm{i}$

APPLY

3 $(2-3\mathrm{i})^2$.

4 $(5-\mathrm{i})(5+\mathrm{i})$.

5 $\dfrac{10}{1+3\mathrm{i}}$.

I.3 Equal complex numbers

Property 2 *equal complex numbers*

$a + b\mathrm{i} = c + d\mathrm{i}$ if and only if $a = c$ **and** $b = d$. One complex equation is two real equations: to find an unknown z, write $z = x + y\mathrm{i}$, expand, and **equate real and imaginary parts**.

Example 4

Find z such that $z + 2z^* = 6 - 4\mathrm{i}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Write $z = x + y\mathrm{i}$	$x + y\mathrm{i} + 2(x - y\mathrm{i}) = 6 - 4\mathrm{i}$
Step 2 Collect	$3x - y\mathrm{i} = 6 - 4\mathrm{i}$
Step 3 Equate real and imaginary parts	$3x = 6,\ -y = -4$
Step 4 Solve	$z = 2 + 4\mathrm{i}$

APPLY

6 $(x + y\mathrm{i})(1 + \mathrm{i}) = 3 + 5\mathrm{i}$, with $x, y \in \mathbb{R}$.

II THE COMPLEX PLANE, SQUARE ROOTS AND CONJUGATE ROOTS

AHL 1.12, 1.14 · Argand diagram, $|z|$, $\arg z$, conjugate pairs

II.1 Modulus and argument

Definition 2 *modulus and argument*

In the **complex plane** (Argand diagram) $z = a + b\mathrm{i}$ is the point (a, b). Its **modulus** $|z| = \sqrt{a^2 + b^2}$ is its distance from 0; its **argument** $\arg z$ is the angle from the positive real axis, taken in $-\pi < \arg z \leq \pi$. z^* is the reflection of z in the real axis, and $zz^* = |z|^2$.

Example 5

Find the modulus and the argument of $z = -1 - \sqrt{3}\,\mathrm{i}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Sketch: which quadrant?	$(-1, -\sqrt{3})$: third quadrant
Step 2 Modulus	$\lvert z\rvert = \sqrt{1+3} = 2$
Step 3 Argument from the basic angle	$\arctan\sqrt{3} = \frac{\pi}{3}$, $\arg z = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$
In the third quadrant the argument is negative, between $-\pi$ and $-\frac{\pi}{2}$.	

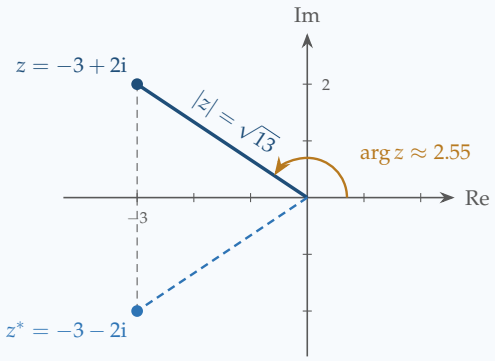


Figure 1 — Exam-style question 2: the root $z = -3 + 2\mathrm{i}$ in the complex plane: $|z| = \sqrt{13}$, and $\arg z = \pi - \arctan\frac{2}{3} \approx 2.55$ because z is in the second quadrant. z^* is its mirror image.

APPLY

1 $|4 - 3\mathrm{i}|$ and $\arg(4 - 3\mathrm{i})$.

2 $|z|$ and $\arg z$ for $z = -\sqrt{3} + \mathrm{i}$.

II.2 Square roots of a complex number

Property 3 *square roots of a complex number*

To find $\sqrt{w}$ with $w = p + q\mathrm{i}$, write $(a + b\mathrm{i})^2 = w$ and equate parts:

$$a^2 - b^2 = p, \qquad 2ab = q.$$

Substituting $b = \frac{q}{2a}$ gives a quadratic in a^2; keep $a^2 > 0$. There are always **two** square roots, $\pm(a + b\mathrm{i})$, opposite each other.

Example 6

Find the square roots of $8 - 6\mathrm{i}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Square and equate parts	$(a + b\mathrm{i})^2 = 8 - 6\mathrm{i}$: $a^2 - b^2 = 8,\ 2ab = -6$
Step 2 Substitute $b = -\frac{3}{a}$	$a^2 - \dfrac{9}{a^2} = 8,\ a^4 - 8a^2 - 9 = 0$
Step 3 Keep $a^2 > 0$	$(a^2 - 9)(a^2 + 1) = 0$: $a^2 = 9, a = \pm 3, b = \mp 1$
Step 4 The two roots	$\pm(3 - \mathrm{i})$

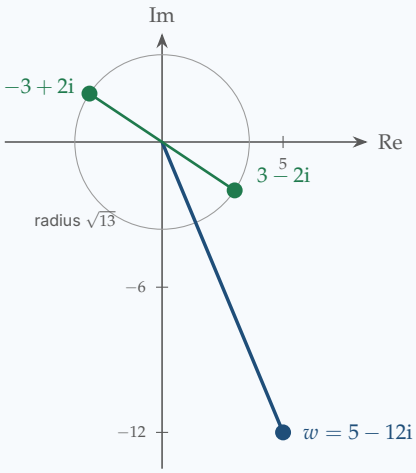


Figure 2 — Exam-style question 2 (a): the two square roots of $w = 5 - 12\mathrm{i}$: $\pm(3 - 2\mathrm{i})$, opposite each other, each at distance $\sqrt{13} = \sqrt{|w|}$ from 0.

APPLY

3 The square roots of $3 + 4\mathrm{i}$.

4 The square roots of $-21 - 20\mathrm{i}$.

II.3 Quadratics and conjugate pairs

Property 4 *conjugate roots*

The quadratic formula works with a negative discriminant: $z^2 - 4z + 13 = 0$ gives $z = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3\mathrm{i}$. If a polynomial has **real coefficients**, its non-real roots come in **conjugate pairs**: if $1 + 2\mathrm{i}$ is a root, so is $1 - 2\mathrm{i}$, and $(z-1)^2 + 4 = z^2 - 2z + 5$ is a factor.

Example 7

Solve $z^2 - 6z + 13 = 0$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Discriminant	$36 - 52 = -16$
Step 2 Its square root	$\sqrt{-16} = 4\mathrm{i}$
Step 3 Quadratic formula	$z = \dfrac{6 \pm 4\mathrm{i}}{2} = 3 \pm 2\mathrm{i}$

Example 8

$z^3 - 3z^2 + z + 5 = 0$ has the root $2 - \mathrm{i}$. Find the other roots.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Real coefficients: the conjugate is a root	$2 + \mathrm{i}$
Step 2 The quadratic factor	$(z-2+\mathrm{i})(z-2-\mathrm{i}) = (z-2)^2 + 1 = z^2 - 4z + 5$
Step 3 The other factor	$z^3 - 3z^2 + z + 5 = (z^2 - 4z + 5)(z + 1)$
Step 4 The roots	$2 \pm \mathrm{i}$ and -1

Watch out 2

Conjugate pairs need real coefficients. In $z^2 + 3z + (1 + 3\mathrm{i}) = 0$ the roots $-\mathrm{i}$ and $-3 + \mathrm{i}$ are not conjugate.

APPLY

5 Solve $z^2 + 4z + 8 = 0$.

6 A real quadratic $z^2 + bz + c$ with the root $1 - 3\mathrm{i}$.

7 A real cubic $z^3 + \cdots$ with the roots 4 and i.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A The Cartesian form

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 6]

The complex number z satisfies $2z + \mathrm{i}z^* = 3 + 4\mathrm{i}$.

(a) Find z, giving your answer in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$. [4]

(b) Hence find z^2 in the form $c + d\mathrm{i}$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Write $z = a + b\mathrm{i}$, expand	$z = a + b\mathrm{i}$: $2a + 2b\mathrm{i} + \mathrm{i}(a - b\mathrm{i}) = (2a + b) + (a + 2b)\mathrm{i}$	M1
	Substituting z and z^* and expanding.	
Step 2 Equate real and imaginary parts	$2a + b = 3$ and $a + 2b = 4$	M1A1
	M1 for equating real and imaginary parts.	
Step 3 Solve	$z = \frac{2}{3} + \frac{5}{3}\mathrm{i}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Square	$\left(\frac{2}{3} + \frac{5}{3}\mathrm{i}\right)^2 = \frac{4}{9} + \frac{20}{9}\mathrm{i} - \frac{25}{9}$	(M1)
Step 2 Simplify	$= -\frac{7}{3} + \frac{20}{9}\mathrm{i}$	A1

B Square roots and conjugate roots

EXAM-STYLE QUESTION 2

Paper 1 · no calculator · [Maximum mark: 7]

Let $w = 5 - 12\text{i}$.

(a) Find the two square roots of w, giving your answers in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$. [4]

(b) Hence solve the equation $z^2 + 3z + 1 + 3\text{i} = 0$. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Square and equate parts	$(a + b\text{i})^2 = 5 - 12\text{i}$: $a^2 - b^2 = 5,\ 2ab = -12$ *M1 for equating parts.*	M1A1
Step 2 Quadratic in a^2	$a^2 - \dfrac{36}{a^2} = 5,\ a^4 - 5a^2 - 36 = 0,\ a^2 = 9$	(M1)
Step 3 The two roots	$3 - 2\text{i}$ and $-3 + 2\text{i}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Quadratic formula	$z = \dfrac{-3 \pm \sqrt{9 - 4(1 + 3\text{i})}}{2} = \dfrac{-3 \pm \sqrt{5 - 12\text{i}}}{2}$ *The discriminant is w: "hence".*	M1
Step 2 Use the root from (a)	$z = \dfrac{-3 \pm (3 - 2\text{i})}{2}$	(A1)
Step 3 The two solutions	$z = -\text{i}$ or $z = -3 + \text{i}$	A1

EXAM-STYLE QUESTION 3

Paper 1 · no calculator · [Maximum mark: 6]

Consider the equation $z^3 + pz^2 + qz - 15 = 0$, where $p, q \in \mathbb{R}$. One root of the equation is $1 + 2\text{i}$.

(a) Write down a second root of the equation. [1]

(b) Find the value of p, the value of q and the third root. [5]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Real coefficients: the conjugate	$1 - 2\text{i}$ *The coefficients are real.*	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The quadratic factor	$(z - 1 - 2\text{i})(z - 1 + 2\text{i}) = z^2 - 2z + 5$	M1A1
Step 2 The third root	$(z^2 - 2z + 5)(z - 3)$, third root 3 *From the constant term: $5 \times (-3) = -15$.*	A1
Step 3 Expand, compare	$z^3 - 5z^2 + 11z - 15$: $p = -5,\ q = 11$	A1A1

C On the GDC (Paper 2)

EXAM-STYLE QUESTION 4 Paper 2 · calculator · [Maximum mark: 7]

Let $z_1 = 3 + 4\mathrm{i}$ and $z_2 = 1 - 2\mathrm{i}$.

(a) Find $\dfrac{z_1}{z_2}$, giving your answer in the form $a + b\mathrm{i}$. [2]

(b) Find $|z_1 z_2|$ and $\arg(z_1 z_2)$. [3]

(c) Find the real number k such that $z_1 + kz_2$ is purely imaginary. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Multiply by the conjugate	$\dfrac{(3+4\mathrm{i})(1+2\mathrm{i})}{5}$	(M1)
Step 2 Compute	$= -1 + 2\mathrm{i}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The product	$z_1 z_2 = 11 - 2\mathrm{i}$	(A1)
Step 2 Modulus	$\lvert z_1 z_2 \rvert = \sqrt{125} = 11.2$	A1
	Or $\lvert z_1 \rvert \lvert z_2 \rvert = 5\sqrt{5}$.	
Step 3 Argument	$\arg(z_1 z_2) = -0.180$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Real part $= 0$	$\mathrm{Re} = 3 + k = 0$	(M1)
Step 2 Solve	$k = -3$	A1

IV WHICH METHOD? READ THE QUESTION

AHL 1.12 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"find z" in an equation with z and z^*	$z = a + b\mathrm{i}$, $z^* = a - b\mathrm{i}$	expand, then equate real and imaginary parts P2
a quotient $\frac{u}{v}$	multiply by v^*	vv^* is real: then split into $a + b\mathrm{i}$ P1
"Re", "Im"	real numbers	no i in $\mathrm{Im}(z)$ W1
"modulus", "argument"	$\lvert z\rvert = \sqrt{a^2 + b^2}$ $-\pi < \arg z \leq \pi$	a sketch first: which quadrant? D2
"the square roots of w"	$(a + b\mathrm{i})^2 = w$	two real equations, a quadratic in a^2, two opposite roots P3
"real coefficients", one root given	conjugate pairs	the quadratic factor $(z - \alpha)(z - \alpha^*)$ P4

In the exam 1 *command terms*

Write down: the conjugate root needs no working. **Find** z **in the form** $a + b\mathrm{i}$: real a and b, simplified. **Hence**: here, recognise the square root just found inside the formula.

V ON THE GDC: PAPER 2

AHL 1.12, 1.14 · NumWorks · every calculator move of this topic

1 Complex arithmetic on the GDC

Settings → Complex format → a+ib → Calculation

Set the complex format to $a + ib$ once; then i is typed with the shift key. **On paper:** the product or quotient set up, then the answer in the form asked.

CALCULATION

```
(3+4i)/(1-2i)
-1+2i
(3+4i)*(1-2i)
11-2i
```

2 Modulus and argument

Calculation → Toolbox → Complex numbers → abs(z), arg(z)

The calculator gives the principal argument in radians (angle unit set to radians). **On paper:** the value to three significant figures.

CALCULATION

```
abs(11-2i)
11.18033989
arg(11-2i)
-0.1798534998
```

ANSWERS TO THE APPLY QUESTIONS

I.1 $4;\ 7$

I.2 $1+3\mathrm{i}$

I.3 $-5-12\mathrm{i}$

I.4 26

I.5 $1-3\mathrm{i}$

I.6 $x-y=3, x+y=5$: $x=4, y=1$

II.1 $5;\ -0.644$

II.2 $2;\ \frac{5\pi}{6}$

II.3 $\pm(2+\mathrm{i})$

II.4 $\pm(2-5\mathrm{i})$

II.5 $-2\pm 2\mathrm{i}$

II.6 $z^2-2z+10$

II.7 $(z-4)(z^2+1)=z^3-4z^2+z-4$

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