# Counting and the extended binomial theorem

HL AA · Proof and complex numbers · AHL 1.10

Counting is multiplying the number of choices at each step, then dividing when the order does not matter. The same $\binom{n}{r}$ that counts choices gives the coefficients of $(1+x)^n$; at HL the power $n$ may also be negative or a fraction, and the expansion becomes an infinite series that is only valid for small $x$.

## I COUNTING PRINCIPLES

AHL 1.10 · product rule, $n!$, ${}^nP_r$, ${}^nC_r$, restrictions

### I.1 The product principle

**Definition 1** *the product principle*

If a first choice can be made in $m$ ways and then, whatever it was, a second in $n$ ways, the two together can be made in $m \times n$ ways. **Or** between two separate cases means **add**; **and then** means **multiply**. In particular, $n$ different objects can be put in a row in $n! = n(n-1)\cdots 2 \times 1$ ways.

**Example 1**

A password is one letter (from 26) followed by three different digits. How many passwords are there?

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Count each place in turn | 26, then 10, 9, 8 |
| **Step 2** Multiply | $26 \times 10 \times 9 \times 8 = 18\,720$ |

**APPLY**

1 A meal: one of 4 starters, 6 mains, 3 desserts.

2 Seven different books in a row.

## I.2 Permutations and combinations

**Definition 2** *permutations and combinations*

An ordered selection of $r$ objects from $n$ different objects is a **permutation**; an unordered one is a **combination**:

$$ {}^nP_r = \frac{n!}{(n-r)!}, \qquad {}^nC_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{{}^nP_r}{r!}. $$

Order matters for a line, a code, a podium; it does not for a team, a committee, a hand of cards.

**Example 2**

Twelve runners race. In how many ways can the gold, silver and bronze medals go? In how many ways can three of them be picked for a relay squad?

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Podium: order matters | ${}^{12}P_3 = 12 \times 11 \times 10 = 1320$ |
| **Step 2** Squad: order does not matter | $\binom{12}{3} = \dfrac{1320}{3!} = 220$ |

**Watch out 1**

Not required in the exam: arrangements of identical objects and arrangements round a circle. Before counting, decide whether order matters; a team chosen with a captain is a combination times a choice of captain.

APPLY

**3** A committee of 4 from 9 people.

**4** A code of 3 different letters from A to H.

**5** A team of 5 from 8, then a captain.

## I.3 Next to each other: one block

**Property 1** *restrictions*

**Together**: glue the objects into one block, arrange the block with the rest, then multiply by the arrangements inside the block. **Never next to each other**: arrange the others first, then place the restricted objects in the gaps between and at the ends. **At least one**: often the total minus the number with none.

**Example 3**

Six different books, three of them on maths, are put on a shelf. In how many ways, if the maths books stay together?

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Glue into one block | 1 block + 3 books = 4 objects |
| **Step 2** Arrange the objects | $4! = 24$ |
| **Step 3** Arrange inside the block | $3! = 6$ |
| **Step 4** Multiply | $24 \times 6 = 144$ |

APPLY

**6** Seven people in a row, two friends side by side.

## I.4 Never next to each other: the gaps

**Example 4**

Four boys and three girls, all different, stand in a row with no two girls next to each other. How many rows?

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Arrange the others first | boys: $4! = 24$ |
| **Step 2** Count the gaps | between and at the ends: 5 gaps |
| **Step 3** Place the girls in the gaps, in order | ${}^5P_3 = 60$ |
| **Step 4** Multiply | $24 \times 60 = 1440$ |

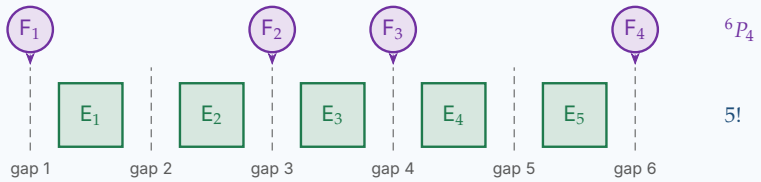



*Figure 1 — Exam-style question 1 (c): five evergreen shrubs make six gaps (green). Four flowering shrubs go into four different gaps, in order: ${}^6P_4 = 360$ ways for each of the 5! rows of evergreens.*

APPLY

**7** Five different red flags and two different blue flags in a row, blues apart.

## I.5 At least one: the complement

**Example 5**

A committee of 4 is chosen from 5 men and 6 women. How many committees contain at least one man?

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Total, with no restriction | $\binom{11}{4} = 330$ |
| **Step 2** None: only women | $\binom{6}{4} = 15$ |
| **Step 3** Total minus none | $330 - 15 = 315$ |

APPLY

**8** A group of 3 from 4 seniors and 6 juniors, at least one senior.

# II THE BINOMIAL THEOREM FOR ANY RATIONAL POWER

AHL 1.10 · $(1+x)^n$, $n \in \mathbb{Q}$, validity $|x| < 1$

## II.1 $(1+x)^n$ for a rational $n$

**Property 2** *the extended binomial theorem*

For any rational $n$ and $|x| < 1$,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

When $n$ is a positive integer the series stops after $n+1$ terms (SL 1.9). Otherwise it never stops: the terms used give an **approximation**, and the series is only valid for $|x| < 1$.

**Example 6**

Expand $(1+x)^{3/2}$ up to the term in $x^3$, and state where it is valid.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** The formula with $n = \frac{3}{2}$, each factor in brackets | $1 + \frac{3}{2}x + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{3!}x^3$ |
| **Step 2** Simplify | $1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3$ |
| **Step 3** Validity | $\lvert x\rvert < 1$ |

**APPLY**

1 $(1+x)^{1/4}$ up to $x^2$.

2 $(1+x)^{-3}$ up to $x^2$.

## II.2 Validity and approximation

**Example 7**

Expand $(1-x)^{-1}$ up to $x^3$ and use it to approximate $\dfrac{1}{0.96}$.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Expand with $n = -1$, $-x$ in brackets | $1 + x + x^2 + x^3$ |
| **Step 2** Choose $x$ inside $\lvert x\rvert < 1$ | $1 - x = 0.96$: $x = 0.04$ |
| **Step 3** Substitute | $1 + 0.04 + 0.0016 + 0.000064 = 1.041664$ |
| The true value is $1.0416666\ldots$ | |

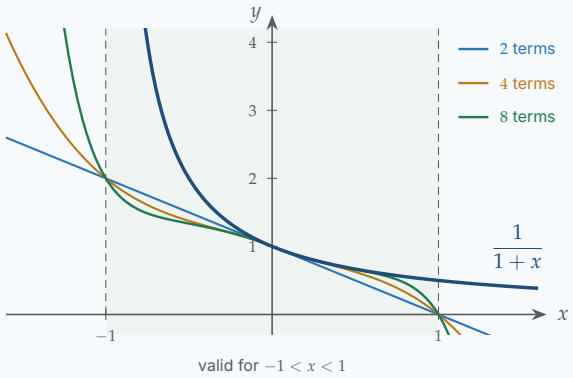



*Figure 2 — Example 7, with $-x$ for $x$: $(1+x)^{-1}$ and its partial sums $1 - x + x^2 - \cdots$ with 2, 4 and 8 terms: inside $-1 < x < 1$ they close in on the curve; outside, they move away from it.*

**APPLY**

**3** Use $(1+x)^{1/3} \approx 1 + \frac{x}{3} - \frac{x^2}{9}$ to approximate $\sqrt[3]{1.03}$.

**4** Can the series of $(1-x)^{-1}$ with $x = 3$ approximate $-\frac{1}{2}$?

## II.3 A bracket that does not start with 1

**Property 3** *a bracket that does not start with 1*

Take out the first term first:

$$(a+bx)^n = a^n \left(1 + \frac{b}{a}x\right)^n, \qquad \text{valid for } \left|\frac{b}{a}x\right| < 1, \text{ that is } |x| < \left|\frac{a}{b}\right|.$$

The factor $a^n$ multiplies every term: $(4+x)^{1/2} = 2\left(1+\frac{x}{4}\right)^{1/2}$.

**Example 8**

Expand $(8-x)^{1/3}$ up to $x^2$, and state where it is valid.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Take out $a^n$ | $8^{1/3}\left(1-\frac{x}{8}\right)^{1/3} = 2\left(1-\frac{x}{8}\right)^{1/3}$ |
| **Step 2** Expand the bracket | $1 + \frac{1}{3}\left(-\frac{x}{8}\right) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(-\frac{x}{8}\right)^2 = 1 - \frac{x}{24} - \frac{x^2}{576}$ |
| **Step 3** Multiply by $a^n$ | $2 - \frac{x}{12} - \frac{x^2}{288}$ |
| **Step 4** Validity | $\left\|\frac{x}{8}\right\| < 1$: $\|x\| < 8$ |

**Watch out 2**

The coefficients are $\dfrac{n(n-1)\cdots}{r!}$, not $\binom{n}{r}$ read from Pascal: Pascal's triangle has no row $-\frac{1}{2}$. Signs alternate as soon as $n$ or $x$ is negative: write every factor in brackets.

**APPLY**

**5** $(2+x)^{-1}$ up to $x^2$.

# III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

## A Counting

**EXAM-STYLE QUESTION 1** Paper 1 · no calculator · [Maximum mark: 7]

A gardener plants nine different shrubs in a straight row: four flowering shrubs and five evergreen shrubs. Find the number of possible rows in which

(a) the four flowering shrubs are next to each other; [2]

(b) the row starts and ends with an evergreen shrub; [2]

(c) no two flowering shrubs are next to each other. [3]

**WORKING (a)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Glue into one block | $6! \times 4!$<br>*The block of four and the five evergreens: six objects.* | (M1) |
| **Step 2** Compute | $= 17\,280$ | A1 |

**WORKING (b)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Fill the ends first | $5 \times 4 \times 7!$<br>*Evergreens at both ends, then the seven others in any order.* | (M1) |
| **Step 2** Compute | $= 100\,800$ | A1 |

**WORKING (c)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Arrange the others first | 5! rows of evergreens | (M1) |
| **Step 2** Place in the gaps, in order | six gaps: ${}^6P_4 = 360$<br>*Or* $\binom{6}{4} \times 4!$. | (M1) |
| **Step 3** Multiply | $120 \times 360 = 43\,200$ | A1 |

## B The extended binomial theorem

**EXAM-STYLE QUESTION 2** Paper 1 · no calculator · [Maximum mark: 7]

Consider the function $f(x) = \sqrt{1+4x}$.

(a) Find the binomial expansion of $f(x)$ in ascending powers of $x$, up to and including the term in $x^3$. [3]

(b) State the set of values of $x$ for which this expansion is valid. [1]

(c) By choosing a suitable value of $x$, use your answer to part (a) to find an approximation to $\sqrt{26}$, correct to five decimal places. [3]

**WORKING (a)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** The formula with $n = \frac{1}{2}$ | $1 + \frac{1}{2}(4x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(4x)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(4x)^3$ *The general form with $n = \frac{1}{2}$ and $4x$ in brackets.* | (M1) |
| **Step 2** Simplify | $= 1 + 2x - 2x^2 + 4x^3$ *A1 for two correct terms after 1.* | A2 |

**WORKING (b)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Validity: $\lvert 4x\rvert < 1$ | $\lvert 4x\rvert < 1,\ -\frac{1}{4} < x < \frac{1}{4}$ | A1 |

**WORKING (c)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Choose $x$ inside the interval | $\sqrt{26} = 5\sqrt{1.04} = 5\,f(0.01)$ *Choosing $x = 0.01$, inside the interval of $(b)$.* | (M1) |
| **Step 2** Substitute | $5(1 + 0.02 - 0.0002 + 0.000004)$ | (A1) |
| **Step 3** Approximation | $\approx 5.09902$ | A1 |

**EXAM-STYLE QUESTION 3** Paper 1 · no calculator · [Maximum mark: 8]

Consider the expression $\dfrac{1+ax}{(2-x)^2}$, where $a \in \mathbb{R}$.

(a) Find the first three terms of the binomial expansion of $(2-x)^{-2}$ in ascending powers of $x$. [4]

(b) State the values of $x$ for which this expansion is valid. [1]

(c) The coefficient of $x^2$ in the expansion of $\dfrac{1+ax}{(2-x)^2}$ is $\dfrac{1}{2}$. Find the value of $a$. [3]

**WORKING (a)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Take out $a^n$ | $(2-x)^{-2} = 2^{-2}\left(1-\frac{x}{2}\right)^{-2}$ | (M1) |
| | *Taking out the 2 first.* | |
| **Step 2** Expand with $n=-2$ | $\frac{1}{4}\left(1+(-2)\left(-\frac{x}{2}\right)+\frac{(-2)(-3)}{2!}\left(-\frac{x}{2}\right)^2\right)$ | (M1) |
| **Step 3** Simplify | $= \frac{1}{4}+\frac{1}{4}x+\frac{3}{16}x^2$ | A2 |
| | *A1 for two correct terms.* | |

**WORKING (b)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Validity | $\lvert x\rvert < 2$ | A1 |

**WORKING (c)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Collect the $x^2$ terms | $\frac{3}{16} + a \times \frac{1}{4}$ | (M1) |
| | *Two ways to make $x^2$: $1 \times \frac{3}{16}x^2$ and $ax \times \frac{1}{4}x$.* | |
| **Step 2** Set equal to $\frac{1}{2}$ | $\frac{3}{16}+\frac{a}{4}=\frac{1}{2}$ | (A1) |
| **Step 3** Solve | $a = \frac{5}{4}$ | A1 |

## C Counting on Paper 2

**EXAM-STYLE QUESTION 4** Paper 2 · calculator · [Maximum mark: 7]

A choir of six singers is chosen from seven sopranos and eight altos.

(a) Find the number of choirs that contain at least two sopranos. [3]

(b) The choir is chosen at random. Find the probability that it contains exactly three sopranos. [2]

(c) A choir of three sopranos and three altos stands in a line, sopranos and altos alternating. Find the number of possible lines. [2]

**WORKING (a)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Total | $\binom{15}{6} = 5005$ in total | (A1) |
| **Step 2** None, then exactly one | none: $\binom{8}{6} = 28$; one: $7 \times \binom{8}{5} = 392$ | (M1) |
| | *The complement: no soprano or exactly one.* | |
| **Step 3** Total minus the rest | $5005 - 28 - 392 = 4585$ | A1 |

**WORKING (b)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Favourable over total | $\dfrac{\binom{7}{3}\binom{8}{3}}{\binom{15}{6}} = \dfrac{35 \times 56}{5005}$ | (M1) |
| **Step 2** Compute | $= 0.392$ | A1 |
| | $\frac{56}{143}$. | |

**WORKING (c)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Alternate, then arrange each group | $2 \times 3! \times 3!$ | (M1) |
| | *Start with a soprano or with an alto.* | |
| **Step 2** Compute | $= 72$ | A1 |

# IV WHICH METHOD? READ THE QUESTION

AHL 1.10 · the whole topic on one page

| THE QUESTION SAYS | YOU KNOW | YOU WRITE |
|---|---|---|
| *"arranged in a row", "a code", "in order"* | order matters<br>$n!$, ${}^{n}P_{r}$ | multiply the choices place by place D1 |
| *"a team", "a committee", "chosen"* | order does not matter<br>$\binom{n}{r}$ | one $\binom{n}{r}$ per group, multiplied D2 |
| *"next to each other"* | one block | arrange the block with the rest, $\times$ inside P1 |
| *"no two next to each other"* | the gaps | others first, then ${}^{\text{gaps}}P_{k}$ P1 |
| *"at least one"* | the complement | total minus none P1 |
| $(1+x)^n$ *with* $n<0$ *or* $n\notin\mathbb{Z}$ | $1+nx+\frac{n(n-1)}{2!}x^2+\cdots$ | each factor in brackets, then validity $\lvert x\rvert<1$ P2 |
| $(a+bx)^n$ | $a^n\left(1+\frac{b}{a}x\right)^n$ | factor first; valid for $\lvert x\rvert<\lvert\frac{a}{b}\rvert$ P3 |
| *"approximate* $\sqrt{k}$*"* | a value of $x$ inside the interval | substitute, then scale back P2 |

**In the exam 1** *command terms*

**Find the number of ways**: show the product that counts them. **State the values of $x$ for which ... is valid**: an inequality in $x$, from $|\ldots|<1$. **Hence**: use the expansion just found.

# V ON THE GDC: PAPER 2

AHL 1.10 · NumWorks · every calculator move of this topic

## 1 nPr, nCr and n! on the GDC

Calculation → Toolbox → Probability → Combinatorics

The NumWorks writes ${}^{n}P_{r}$ as `permute(n,k)` and $\binom{n}{r}$ as `binomial(n,k)`; $n!$ is in the same menu. **On paper:** the product that counts, then the value.

CALCULATION

```
permute(6,4)
360
binomial(15,6)
5005
5!*permute(6,4)
43200
```

### 2 An expansion checked numerically

Calculation → (1+4*0.01)^0.5

Compare the series with the exact value: the difference is the size of the next term. **On paper:** the substitution into the series, never only the calculator value.

**CALCULATION**

```
(1+4*0.01)^0.5
1.019803903
1+2*0.01-2*0.01^2+4*0.01^3
1.019804
```

## ANSWERS TO THE APPLY QUESTIONS

**I.1** $72$

**I.2** $7! = 5040$

**I.3** $126$

**I.4** ${}^{8}P_{3} = 336$

**I.5** $56 \times 5 = 280$

**I.6** $6! \times 2 = 1440$

**I.7** $5! \times {}^{6}P_{2} = 3600$

**I.8** $120 - 20 = 100$

**II.1** $1 + \frac{x}{4} - \frac{3x^2}{32},\ |x| < 1$

**II.2** $1 - 3x + 6x^2,\ |x| < 1$

**II.3** $1.0099$

**II.4** No: $x = 3$ is outside $|x| < 1$

**II.5** $\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8},\ |x| < 2$

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