De Moivre's theorem, powers and roots

HL AA · Proof and complex numbers · AHL 1.14

Since multiplying adds the arguments, the nth power of $re^{i\theta}$ turns n times by θ: that is De Moivre's theorem. Read backwards, it finds the nth roots of any complex number: n points equally spaced on a circle. Read with cos and sin, it turns $\cos 3\theta$ into a polynomial in $\cos\theta$.

I DE MOIVRE'S THEOREM AND POWERS

AHL 1.14 · $(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$

I.1 Powers with De Moivre's theorem

Property 1 *De Moivre's theorem*

For every integer n,

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta), \qquad \left(re^{i\theta}\right)^n = r^n e^{in\theta}.$$

For $n \in \mathbb{Z}^+$ it is proved by induction (sub-notion *Proof by induction and contradiction*); it also holds for rational n, which gives the roots.

Example 1

Find $\left(\sqrt{3} - i\right)^6$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Euler form first	$\sqrt{3} - i = 2e^{-i\pi/6}$
Step 2 Power the modulus, multiply the argument	$2^6 e^{-i\pi}$
Step 3 Back to $a + bi$	$64(-1) = -64$

Watch out 1

$(1+i)^8$ is not $1 + i^8$: power the modulus and multiply the argument, $(\sqrt{2})^8 e^{i2\pi} = 16$.

APPLY

1 $(2+2i)^3$ in Cartesian form.

2 $\left(2e^{i\pi/8}\right)^4$.

I.2 Powers that are real or purely imaginary

Property 2 *powers that are real or imaginary*

$\left(re^{i\theta}\right)^n$ is **real** when $n\theta = k\pi$ and **purely imaginary** when $n\theta = \frac{\pi}{2} + k\pi$, with $k \in \mathbb{Z}$: for the smallest positive n, try $k = 0, 1, 2, \ldots$

Example 2

$z = 1.1e^{i\pi/6}$. Find the smallest positive integer n for which z^n is real, and the value of z^n.

WHAT YOU DO	WHAT YOU WRITE
Step 1 De Moivre	$z^n = 1.1^n e^{in\pi/6}$
Step 2 Real: $n\theta = k\pi$	$\frac{n\pi}{6} = k\pi$: $n = 6k$
Step 3 Smallest n	$k = 1$: $n = 6$
Step 4 The value	$z^6 = 1.1^6 e^{i\pi} = -1.771561$

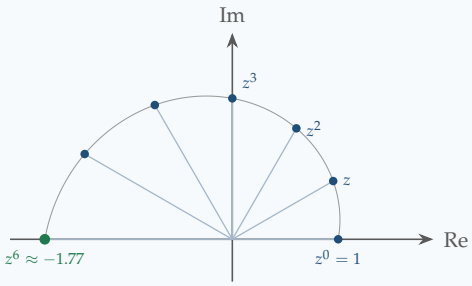


Figure 1 — Example 2: the powers of $z = 1.1e^{i\pi/6}$. Each one is the previous one turned by $\frac{\pi}{6}$ and stretched by 1.1, so the points lie on a spiral; z^6 is real and negative.

APPLY

3 $z = 3e^{i3\pi/10}$: the smallest n with z^n purely imaginary, and z^n.

4 $z = \sqrt{2}\,e^{i2\pi/9}$: the smallest n with z^n real, and z^n.

I.3 Multiple angles

Property 3 *multiple angles*

Expanding the left side of $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ with the binomial theorem and equating real parts gives

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta.$$

Example 3

With $c = \cos\theta$ and $s = \sin\theta$, use $(c + is)^2$ to express $\cos 2\theta$ and $\sin 2\theta$ in terms of c and s.

WHAT YOU DO	WHAT YOU WRITE
Step 1 De Moivre	$(c + is)^2 = \cos 2\theta + i\sin 2\theta$
Step 2 Expand the left side	$c^2 + 2ics + i^2s^2 = (c^2 - s^2) + i(2cs)$
Step 3 Equate real and imaginary parts	$\cos 2\theta = c^2 - s^2$, $\sin 2\theta = 2cs$

APPLY

5 $\sin 4\theta$ in terms of c and s.

II ROOTS OF COMPLEX NUMBERS

AHL 1.14 · $z^n = w$: n roots on a circle, $\frac{2\pi}{n}$ apart

II.1 The nth roots

Property 4 *the nth roots*

If $w = Re^{i\varphi}$, the equation $z^n = w$ has exactly n roots

$$z_k = R^{1/n}\, e^{i\frac{\varphi+2k\pi}{n}}, \qquad k = 0, 1, \ldots, n-1.$$

They lie on the circle of radius $R^{1/n}$, $\frac{2\pi}{n}$ apart: the vertices of a regular n-gon. Their sum is 0.

Example 4

Solve $z^4 = -81$, giving the roots in the form $re^{i\theta}$, $-\pi < \theta \leq \pi$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Euler form of the right side	$-81 = 81e^{i\pi}$
Step 2 Modulus of the roots	$81^{1/4} = 3$
Step 3 Arguments: $\frac{\varphi+2k\pi}{n}$, $k = 0$ to 3	$\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$
Step 4 Write them in $(-\pi, \pi]$	$3e^{i\pi/4}$, $3e^{i3\pi/4}$, $3e^{-i3\pi/4}$, $3e^{-i\pi/4}$

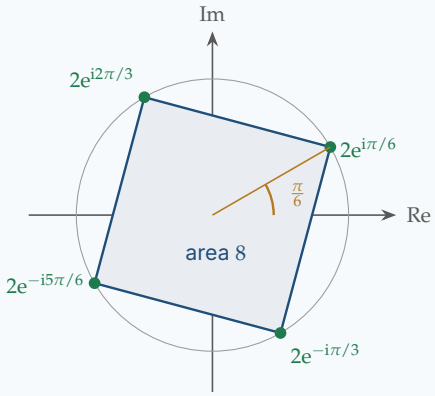


Figure 2 — Exam-style question 3: the four roots of $z^4 = -8 + 8\sqrt{3}\,i = 16e^{i2\pi/3}$. Radius 2, first argument $\frac{2\pi/3}{4} = \frac{\pi}{6}$, then a quarter turn each time. They form a square of area 8.

Watch out 2

Write the arguments in the form asked: $\frac{7\pi}{6}$ is $-\frac{5\pi}{6}$ when $-\pi < \theta \leq \pi$. And there are exactly n roots: listing $k = 0$ to n repeats the first.

APPLY

1 Solve $z^2 = 4i$.

2 Solve $z^5 = 1$.

3 The area of the hexagon made by the roots of $z^6 = 64$.

II.2 From one root to all of them

Property 5 *from one root to all of them*

If one root z_0 of $z^n = w$ is known, the others are $z_0 e^{i2k\pi/n}$: multiply by the nth roots of unity, $1,\ e^{i2\pi/n},\ \ldots$

Example 5

Show that $3 + i$ is a root of $z^3 = 18 + 26i$, then find the other two roots.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Check the root	$(3+i)^3 = 27 + 27i - 9 - i = 18 + 26i$
Step 2 The cube roots of unity	$e^{\pm i2\pi/3} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$
Step 3 Multiply	$(3+i)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{-3-\sqrt{3}}{2} + \frac{3\sqrt{3}-1}{2}i$
Step 4 The other one	$(3+i)\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \frac{\sqrt{3}-3}{2} - \frac{1+3\sqrt{3}}{2}i$

APPLY

4 2 is a root of $z^4 = 16$: the other roots.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A Multiple angles

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 7]

Let $c = \cos\theta$ and $s = \sin\theta$.

(a) By considering $(c + \mathrm{i}s)^3$, show that $\cos 3\theta = 4c^3 - 3c$. [4]

(b) Hence solve the equation $8x^3 - 6x - 1 = 0$, giving your answers in the form $\cos\alpha$, where $0 \le \alpha \le \pi$. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 De Moivre	$(c + \mathrm{i}s)^3 = \cos 3\theta + \mathrm{i}\sin 3\theta$ *De Moivre.*	M1
Step 2 Binomial expansion	$= c^3 + 3c^2(\mathrm{i}s) + 3c(\mathrm{i}s)^2 + (\mathrm{i}s)^3$ *The binomial expansion.*	M1
Step 3 Collect real and imaginary parts	$= c^3 - 3cs^2 + \mathrm{i}\left(3c^2 s - s^3\right)$	A1
Step 4 Equate real parts, $s^2 = 1 - c^2$	real parts: $\cos 3\theta = c^3 - 3c\left(1 - c^2\right)$ *$s^2 = 1 - c^2$ must be seen.*	A1
Step 5 Simplify	$= 4c^3 - 3c$	AG

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Substitute $x = \cos\theta$	$x = \cos\theta$: $2\cos 3\theta - 1 = 0$, $\cos 3\theta = \frac{1}{2}$	M1
Step 2 All the angles 3θ	$3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$	(A1)
Step 3 The three roots	$x = \cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}$	A1

B Powers

EXAM-STYLE QUESTION 2 Paper 1 · no calculator · [Maximum mark: 6]

Let $z = \sqrt{3} + \mathrm{i}$.

(a) Write z in the form $re^{\mathrm{i}\theta}$. [2]

(b) Find the smallest positive integer n for which z^n is purely imaginary, and the value of z^n for this n. [4]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Modulus	$r = 2$	A1
Step 2 Argument	$\theta = \frac{\pi}{6}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 De Moivre	$z^n = 2^n e^{\mathrm{i}n\pi/6}$	M1
Step 2 Purely imaginary: $n\theta = \frac{\pi}{2} + k\pi$	$\frac{n\pi}{6} = \frac{\pi}{2} + k\pi$	M1
Step 3 Smallest n	$n = 3$	A1
Step 4 The value of z^n	$z^3 = 8\mathrm{i}$	A1

C Roots

EXAM-STYLE QUESTION 3 Paper 1 · no calculator · [Maximum mark: 7]

Consider the equation $z^4 = -8 + 8\sqrt{3}\,\mathrm{i}$.

(a) Write $-8 + 8\sqrt{3}\,\mathrm{i}$ in the form $re^{i\theta}$. [2]

(b) Solve the equation, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [3]

(c) The roots are the vertices of a quadrilateral in the complex plane. Find its area. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Modulus	$r = \sqrt{64 + 192} = 16$	A1
Step 2 Argument	$\theta = \frac{2\pi}{3}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The root formula	$z = 16^{1/4} e^{i\frac{2\pi/3 + 2k\pi}{4}}$	(M1)
Step 2 The first roots	$2e^{i\pi/6},\ 2e^{i2\pi/3}$	A1
Step 3 The last roots, in $(-\pi, \pi]$	$2e^{-i5\pi/6},\ 2e^{-i\pi/3}$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The polygon	a square with diagonals 4: $\frac{1}{2} \times 4 \times 4$ *Or a square of side $2\sqrt{2}$.*	(M1)
Step 2 Area	$= 8$	A1

D Roots on the GDC (Paper 2)

EXAM-STYLE QUESTION 4 Paper 2 · calculator · [Maximum mark: 8]

Consider the equation $z^3 = 2 + 11\mathrm{i}$.

(a) Show that $2 + \mathrm{i}$ is a root of the equation. [2]

(b) Find the other two roots, giving your answers in the form $a + b\mathrm{i}$. [4]

(c) The three roots form a triangle in the complex plane. Find its area. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Expand the cube	$(2+\mathrm{i})^3 = 8 + 12\mathrm{i} + 6\mathrm{i}^2 + \mathrm{i}^3$	M1
Step 2 Simplify	$= 8 + 12\mathrm{i} - 6 - \mathrm{i} = 2 + 11\mathrm{i}$	A1
	AG: every term seen.	

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Multiply by the cube roots of unity	multiply by $\mathrm{e}^{\pm \mathrm{i}2\pi/3}$	M1
Step 2 Expand	$(2+\mathrm{i})\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right)$	(A1)
Step 3 The two roots	$-1.87 + 1.23\mathrm{i}$, $-0.134 - 2.23\mathrm{i}$	A1A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Equilateral triangle, circumradius	equilateral, circumradius $\lvert 2+\mathrm{i}\rvert = \sqrt{5}$: $3 \times \frac{1}{2}(\sqrt{5})^2 \sin\frac{2\pi}{3}$	(M1)
Step 2 Area	$= 6.50$	A1
	Exact value $\frac{15\sqrt{3}}{4}$.	

IV WHICH METHOD? READ THE QUESTION

AHL 1.14 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"z^n", a power	Euler form first	$r^n\mathrm{e}^{\mathrm{i}n\theta}$, then back to $a+b\mathrm{i}$ if asked P1
"real", "purely imaginary"	$n\theta = k\pi$ $n\theta = \frac{\pi}{2} + k\pi$	smallest n: $k = 0, 1, \ldots$ P2
"$\cos 3\theta$ in terms of $\cos\theta$"	De Moivre + binomial	equate real parts, then $s^2 = 1 - c^2$ P3
"solve $z^n = w$"	n roots, $\frac{2\pi}{n}$ apart	$R^{1/n}\mathrm{e}^{\mathrm{i}(\varphi+2k\pi)/n}$, $k = 0$ to $n-1$ P4
one root given	roots of unity	multiply by $\mathrm{e}^{\mathrm{i}2k\pi/n}$ P5
"the polygon formed by the roots"	regular, centre 0	n triangles $\frac{1}{2}R^{2/n}\sin\frac{2\pi}{n}$ P4

In the exam 1 *command terms*

Show that: every step written, the given answer reached with nothing skipped (AG). **Hence solve**: use the identity just proved. **In the form** $r\mathrm{e}^{\mathrm{i}\theta}$ **with** $-\pi < \theta \leq \pi$: adjust every argument.

V ON THE GDC: PAPER 2

AHL 1.14 · NumWorks · every calculator move of this topic

1 Checking a root on the GDC

Calculation → (2+i)^3

The NumWorks computes powers and products of complex numbers; $\mathrm{e}^{2\pi\mathrm{i}/3}$ turns a root into the next one. **On paper:** the method, then each root to three significant figures.

CALCULATION

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(2+i)^3
2+11i
(2+i)*e^(2πi/3)
-1.866+1.232i
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ANSWERS TO THE APPLY QUESTIONS

I.1 $16\sqrt{2}\,\mathrm{e}^{\mathrm{i}3\pi/4} = -16 + 16\mathrm{i}$

I.2 $16\mathrm{e}^{\mathrm{i}\pi/2} = 16\mathrm{i}$

I.3 $n = 5$; $-243\mathrm{i}$

I.4 $n = 9$; $16\sqrt{2}$

I.5 $4c^3s - 4cs^3$

II.1 $2\mathrm{e}^{\mathrm{i}\pi/4} = \sqrt{2} + \sqrt{2}\,\mathrm{i}$, $2\mathrm{e}^{-\mathrm{i}3\pi/4} = -\sqrt{2} - \sqrt{2}\,\mathrm{i}$

II.2 1, $\mathrm{e}^{\pm\mathrm{i}2\pi/5}$, $\mathrm{e}^{\pm\mathrm{i}4\pi/5}$

II.3 $6 \times \frac{1}{2} \times 2^2 \sin\frac{\pi}{3} = 6\sqrt{3}$

II.4 $2\mathrm{i}$, -2, $-2\mathrm{i}$

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