De Moivre's theorem, powers and roots

Skills sheet · HL AA · AHL 1.14

KEY FACTS

$(re^{i\theta})^n = r^n e^{in\theta}$ P1

Real: $n\theta = k\pi$; imaginary: $n\theta = \frac{\pi}{2} + k\pi$ P2

$\cos 3\theta$: De Moivre, binomial, real parts P3

$z^n = Re^{i\varphi}$: $R^{1/n}e^{i(\varphi+2k\pi)/n}$, $k = 0..n-1$ P4

One root $\times\, e^{i2k\pi/n}$ gives the others P5

$(1+i)^8 \neq 1+i^8$ W1.

Exactly n roots W2.

Arguments in $(-\pi, \pi]$ W2.

WHAT UNLOCKS

- a high power $\rightarrow$ Euler form, power the modulus, multiply the argument.
- "smallest n such that z^n is real" $\rightarrow$ $n\theta = k\pi$, then $k = 1, 2, \ldots$
- "by considering $(c+is)^n$" $\rightarrow$ two expansions of one number, equate parts.
- "solve $z^n = w$" $\rightarrow$ w in Euler form, then n roots $\frac{2\pi}{n}$ apart.

WORKED EXAMPLE ①

Find $(1-i)^6$ in the form $a+bi$. no GDC

$1-i = \sqrt{2}e^{-i\pi/4}$, $(\sqrt{2})^6 e^{-i6\pi/4} = 8e^{-i3\pi/2}$ (M1)

$= 8i$ A1

Method

A power

① Euler form of z.
② r^n and $n\theta$.
③ Back to $a+bi$.

WORKED EXAMPLE ②

Solve $z^3 = 8i$, giving the roots in the form $re^{i\theta}$, $-\pi < \theta \leq \pi$. no GDC

$8i = 8e^{i\pi/2}$, $z = 2e^{i(\pi/2+2k\pi)/3}$ (M1)

$2e^{i\pi/6}$, $2e^{i5\pi/6}$, $2e^{-i\pi/2}$ A1A1

Method

The nth roots

① $w = Re^{i\varphi}$.
② $k = 0$ to $n-1$.
③ $\frac{3\pi}{2}$ becomes $-\frac{\pi}{2}$.

FLUENCY

●○○ **1.** $\left(2e^{i\pi/5}\right)^5$ no GDC

●○○ **2.** $\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)^4$ no GDC

●○○ **3.** $(1+i)^4$ no GDC

●○○ **4.** The modulus of the roots of $z^5 = 32$ no GDC

●○○ **5.** The sum of the roots of $z^6 = 7$ no GDC

●○○ **6.** $\left(e^{i\pi/2}\right)^{-1}$ no GDC

REASONING

●●● **Question 1.** Let $z = 1 + \sqrt{3}\,\mathrm{i}$. [no GDC]

(a) Find z^5, giving your answer in the form $a + b\mathrm{i}$. [3]

●●● **Question 2.** Let $z = -1 + \mathrm{i}$. [no GDC]

(a) Find the smallest positive integer n for which z^n is real, and the value of z^n for this n. [4]

●●● **Question 3.** Consider the equation $z^3 = -27$. [no GDC]

(a) Solve the equation, giving your answers in the form $a + b\mathrm{i}$. [4]

PROBLEMS

●●● **Question 4.** Let $c = \cos\theta$ and $s = \sin\theta$. [no GDC]

(a) By considering $(c + \mathrm{i}s)^4$, show that $\cos 4\theta = 8c^4 - 8c^2 + 1$. [5]

(b) Hence solve the equation $8x^4 - 8x^2 + 1 = 0$, giving your answers in the form $\cos\alpha$, where $0 \leq \alpha \leq \pi$. [3]

●●● **Question 5.** Consider the equation $z^4 = -4$. [no GDC]

(a) Show that $1 - \mathrm{i}$ is a root of the equation. [2]

(b) Hence find the other three roots, in the form $a + b\mathrm{i}$. [3]

(c) Find the area of the polygon whose vertices are the four roots. [2]

ANSWERS

1 $32e^{i\pi} = -32$

2 $e^{i\pi/2} = i$

3 $(\sqrt{2})^4 e^{i\pi} = -4$

4 $32^{1/5} = 2$

5 0

6 $e^{-i\pi/2} = -i$

Q1 $z = 2e^{i\pi/3},\ z^5 = 32e^{i5\pi/3} = 16 - 16\sqrt{3}\,i$

Q2 $z = \sqrt{2}\,e^{i3\pi/4};\ \frac{3n\pi}{4} = k\pi:\ n = 4,\ z^4 = 4e^{i3\pi} = -4$

Q3 $-27 = 27e^{i\pi}:\ 3e^{\pm i\pi/3}, 3e^{i\pi}:\ \frac{3}{2} \pm \frac{3\sqrt{3}}{2}i,\ -3$

Q4 real part $c^4 - 6c^2s^2 + s^4$ with $s^2 = 1 - c^2;\ \cos 4\theta = 0:\ \cos\frac{\pi}{8}, \cos\frac{3\pi}{8}, \cos\frac{5\pi}{8}, \cos\frac{7\pi}{8}$

Q5 $(1-i)^2 = -2i, (-2i)^2 = -4;$ multiply by i: $1+i, -1+i, -1-i;$ a square of side 2: area 4

Written and verified by a practising IB mathematics teacher, produced with AI assistance; every result recomputed by software, every figure computed from its function. Not affiliated with or endorsed by the International Baccalaureate Organization. © 2026 Hence