Polar and Euler forms

A point of the plane can be given by its coordinates, or by its distance from 0 and an angle. Written with the angle, a complex number is $r(\cos\theta + \mathrm{i}\sin\theta) = r\,\mathrm{e}^{\mathrm{i}\theta}$, and multiplication becomes simple: the moduli multiply and the arguments add. That is a rotation and an enlargement of the plane.

I THE MODULUS–ARGUMENT FORM

AHL 1.13 · $z = r(\cos\theta + \mathrm{i}\sin\theta) = r\,\mathrm{cis}\,\theta = r\mathrm{e}^{\mathrm{i}\theta}$

I.1 Polar and Euler forms

Definition 1 *polar and Euler forms*

If $|z| = r$ and $\arg z = \theta$, then

$$z = r(\cos\theta + \mathrm{i}\sin\theta) = r\,\mathrm{cis}\,\theta = r\mathrm{e}^{\mathrm{i}\theta}.$$

The last one is the **Euler form**, from $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$. Back to Cartesian: $a = r\cos\theta$, $b = r\sin\theta$.

Example 1

Write $z = 3\mathrm{e}^{-\mathrm{i}\pi/6}$ in the form $a + b\mathrm{i}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 $a = r\cos\theta$	$3\cos\left(-\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2}$
Step 2 $b = r\sin\theta$	$3\sin\left(-\frac{\pi}{6}\right) = -\frac{3}{2}$
Step 3 Write z	$z = \frac{3\sqrt{3}}{2} - \frac{3}{2}\mathrm{i}$

APPLY

1 $2\mathrm{e}^{\mathrm{i}\pi/4}$ in Cartesian form.

2 $6\mathrm{e}^{-\mathrm{i}\pi/2}$ in Cartesian form.

I.2 Finding the argument

Property 1 *finding the argument*

Sketch z first. Then $\theta = \arctan\left|\frac{b}{a}\right|$ adjusted to the quadrant: first θ, second $\pi - \theta$, third $-(\pi - \theta)$, fourth $-\theta$, so that $-\pi < \arg z \leq \pi$. For example $-\sqrt{3} + \mathrm{i} = 2\mathrm{e}^{\mathrm{i}5\pi/6}$.

Example 2

Write $z = -3\sqrt{3} - 3\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Sketch: which quadrant?	$(-3\sqrt{3}, -3)$: third quadrant
Step 2 Modulus	$r = \sqrt{27 + 9} = 6$
Step 3 Reference angle	$\arctan\dfrac{3}{3\sqrt{3}} = \dfrac{\pi}{6}$
Step 4 Argument in the quadrant	$\theta = -\left(\pi - \frac{\pi}{6}\right) = -\frac{5\pi}{6}$: $z = 6\mathrm{e}^{-\mathrm{i}5\pi/6}$

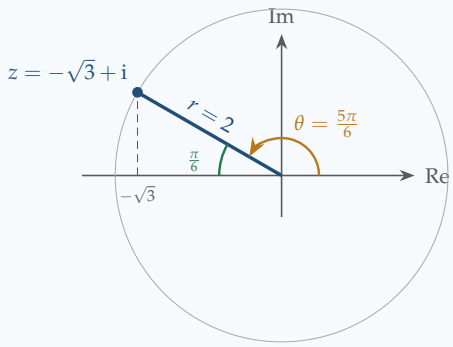


Figure 1 — Exam-style question 1 (a): $z = -\sqrt{3} + \mathrm{i}$, modulus 2, reference angle $\frac{\pi}{6}$, argument $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$ because z is in the second quadrant.

Watch out 1

$\arctan\frac{b}{a}$ alone gives the wrong angle in the second and third quadrants: $\arctan\frac{1}{-\sqrt{3}} = -\frac{\pi}{6}$, but $\arg(-\sqrt{3} + \mathrm{i}) = \frac{5\pi}{6}$. And $r > 0$: $-2\operatorname{cis}\theta$ is not in polar form.

APPLY

3 $2 - 2\sqrt{3}\,\mathrm{i}$ in Euler form.

4 -7 and $4\mathrm{i}$ in Euler form.

II PRODUCTS, QUOTIENTS AND THEIR GEOMETRY

AHL 1.13 · moduli multiply, arguments add; rotation and enlargement

II.1 Products and quotients

Property 2 *products and quotients*

$$r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} = r_1 r_2\, e^{i(\theta_1+\theta_2)}, \qquad \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)}.$$

So $|zw| = |z||w|$ and $\arg(zw) = \arg z + \arg w$, adjusted by $\pm 2\pi$ into $(-\pi, \pi]$.

Example 3

$z = 6e^{i3\pi/4}$ and $w = 2e^{i\pi/2}$. Find zw and $\dfrac{z}{w}$ in the form $re^{i\theta}$, $-\pi < \theta \leq \pi$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Multiply moduli, add arguments	$zw = 12\, e^{i5\pi/4}$
Step 2 Adjust into $(-\pi, \pi]$	$\frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$: $zw = 12\, e^{-i3\pi/4}$
Step 3 Divide moduli, subtract arguments	$\dfrac{z}{w} = 3\, e^{i\pi/4}$

APPLY

1 $4e^{i2\pi/5} \times 3e^{i\pi/5}$.

2 $\dfrac{8e^{-i2\pi/3}}{2e^{i\pi/2}}$.

II.2 Multiplying is rotating

Property 3 *multiplying is rotating*

Multiplying every point by $w = re^{i\alpha}$ **rotates** the plane by α about 0 and **enlarges** it by the factor r. Multiplying by $i = e^{i\pi/2}$ is a quarter turn; adding w is a translation.

Example 4

$z = 2e^{i\pi/6}$ and $w = 1.5e^{i\pi/3}$. Find zw and describe how zw is obtained from z.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Multiply moduli, add arguments	$zw = 3\, e^{i\pi/2} = 3i$
Step 2 Read the geometry from w	rotation by $\frac{\pi}{3}$ about 0, enlargement by 1.5

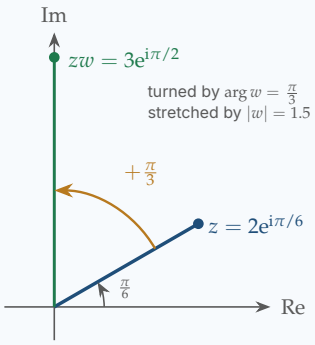


Figure 2 — Example 4: zw is z turned by $\frac{\pi}{3}$ and stretched by 1.5, so $|zw| = 3$ and $\arg(zw) = \frac{\pi}{2}$.

APPLY

3 Multiply $3 + 2\mathrm{i}$ by i: the image, and the transformation.

4 Rotate the point 4 by $\frac{\pi}{3}$ about 0.

II.3 Sums, and exact values from two forms

Property 4 *exact values from two forms*

Computing the same product in Cartesian form and in Euler form gives two expressions of one number: equating them gives exact values such as $\cos\frac{5\pi}{12}$, as in Exam-style question 2.

Example 5

Using $u = 2 + 2\mathrm{i}$ and $v = -1 + \sqrt{3}\,\mathrm{i}$, find the exact value of $\cos\dfrac{11\pi}{12}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 The product in Cartesian form	$uv = -2 + 2\sqrt{3}\,\mathrm{i} - 2\mathrm{i} - 2\sqrt{3} = (-2 - 2\sqrt{3}) + (2\sqrt{3} - 2)\mathrm{i}$
Step 2 Euler forms	$u = 2\sqrt{2}\,e^{i\pi/4}$, $v = 2e^{i2\pi/3}$
Step 3 The product in Euler form	$uv = 4\sqrt{2}\,e^{i11\pi/12}$
Step 4 Equate real parts	$4\sqrt{2}\cos\frac{11\pi}{12} = -2 - 2\sqrt{3}$: $\cos\frac{11\pi}{12} = -\dfrac{\sqrt{6}+\sqrt{2}}{4}$

Watch out 2

Adding complex numbers does not add their moduli or their arguments: for a sum, go back to Cartesian form.

APPLY

5 $z = 2e^{i\pi/2}$, $w = 2$: $z + w$ in Euler form. Is $|z + w| = |z| + |w|$?

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A Polar form, products and quotients

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 7]

Let $z = -\sqrt{3} + \mathrm{i}$ and $w = 2\mathrm{e}^{-\mathrm{i}\pi/4}$.

(a) Find the modulus and the argument of z. [3]

(b) Find zw, giving your answer in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [2]

(c) Find $\dfrac{z}{w}$ in the same form. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Modulus	$\lvert z\rvert = \sqrt{3+1} = 2$	A1
Step 2 Sketch: reference angle	reference angle $\arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$, second quadrant	(M1)
Step 3 Argument in the quadrant	$\arg z = \frac{5\pi}{6}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Multiply moduli, add arguments	$2 \times 2 = 4,\ \frac{5\pi}{6} - \frac{\pi}{4} = \frac{7\pi}{12}$	(M1)
Step 2 Write in Euler form	$zw = 4\mathrm{e}^{\mathrm{i}7\pi/12}$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Divide moduli, subtract arguments	$\frac{2}{2} = 1,\ \frac{5\pi}{6} + \frac{\pi}{4} = \frac{13\pi}{12}$	(M1)
Step 2 Adjust into $(-\pi, \pi]$	$\frac{z}{w} = \mathrm{e}^{-\mathrm{i}11\pi/12}$ $\frac{13\pi}{12} - 2\pi$, *inside* $(-\pi, \pi]$.	A1

B Exact values from two forms

EXAM-STYLE QUESTION 2 Paper 1 · no calculator · [Maximum mark: 7]

Let $u = 1 + \mathrm{i}$ and $v = \sqrt{3} + \mathrm{i}$.

(a) Find uv in the form $a + b\mathrm{i}$. [2]

(b) Write u and v in the form $re^{\mathrm{i}\theta}$. Hence write uv in the same form. [3]

(c) Hence find the exact value of $\cos\dfrac{5\pi}{12}$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Expand	$\sqrt{3} + \mathrm{i} + \sqrt{3}\,\mathrm{i} - 1$	(M1)
Step 2 Collect	$= (\sqrt{3} - 1) + (\sqrt{3} + 1)\mathrm{i}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Euler forms of u and v	$u = \sqrt{2}\,e^{\mathrm{i}\pi/4},\ v = 2e^{\mathrm{i}\pi/6}$	A1A1
Step 2 Multiply	$uv = 2\sqrt{2}\,e^{\mathrm{i}5\pi/12}$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Equate real parts	$2\sqrt{2}\cos\frac{5\pi}{12} = \sqrt{3} - 1$ *Equating the real parts of the two forms.*	M1
Step 2 Exact value	$\cos\frac{5\pi}{12} = \dfrac{\sqrt{3} - 1}{2\sqrt{2}} = \dfrac{\sqrt{6} - \sqrt{2}}{4}$	A1

C On the GDC (Paper 2)

EXAM-STYLE QUESTION 3 Paper 2 · calculator · [Maximum mark: 6]

Let $z = 4e^{1.2i}$ and $w = 3e^{-0.5i}$. The points Z and W represent z and w in the complex plane, and O is the origin.

(a) Find $z + w$ in the form $a + bi$. [2]

(b) Hence write $z + w$ in the form $re^{i\theta}$. [2]

(c) Find the area of triangle OZW. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Convert, then add	$4\cos 1.2 + 3\cos(-0.5) + i(4\sin 1.2 + 3\sin(-0.5))$	(M1)
Step 2 Compute	$= 4.08 + 2.29i$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Modulus	$r = \sqrt{4.08^2 + 2.29^2} = 4.68$	A1
Step 2 Argument	$\theta = \arctan\frac{2.29}{4.08} = 0.511$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The angle ZOW	angle $ZOW = 1.2 - (-0.5) = 1.7$	(M1)
Step 2 Area $\frac{1}{2}ab\sin C$	$\frac{1}{2} \times 4 \times 3\sin 1.7 = 5.95$	A1

IV WHICH METHOD? READ THE QUESTION

AHL 1.13 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"modulus and argument"	$r = \sqrt{a^2 + b^2}$ sketch, quadrant	θ in $(-\pi, \pi]$ P1
"in the form $re^{i\theta}$"	$r > 0$ $-\pi < \theta \leq \pi$	adjust by 2π if needed D1
a product or a quotient	Euler form	multiply/divide moduli, add/subtract arguments P2
a sum $z + w$	Cartesian form	convert, add, convert back W2
"multiplied by i", "rotated"	rotation by $\arg w$	and enlargement by $\lvert w \rvert$ P3
"hence find the exact value of cos*..."*	the same number in two forms	equate real (or imaginary) parts P4

In the exam 1 *command terms*

In the form $re^{i\theta}$, $-\pi < \theta \leq \pi$: both conditions are checked by the markscheme. **Exact value**: surds and π, never a decimal. **Hence**: use the two forms just found.

V ON THE GDC: PAPER 2

AHL 1.13 · NumWorks · every calculator move of this topic

1 Polar form on the GDC

Settings → Complex format → re^(iθ) → Calculation

With the complex format set to $re^{i\theta}$ the NumWorks answers in Euler form directly. **On paper:** the sum in Cartesian form first, then r and θ to three significant figures.

CALCULATION

```
4e^(1.2i)+3e^(-0.5i)
4.681e^(0.511i)
4e^(1.2i)*3e^(-0.5i)
12e^(0.7i)
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ANSWERS TO THE APPLY QUESTIONS

I.1 $\sqrt{2} + \sqrt{2}\,i$

I.2 $-6i$

I.3 $4e^{-i\pi/3}$

I.4 $7e^{i\pi}$; $4e^{i\pi/2}$

II.1 $12e^{i3\pi/5}$

II.2 $4e^{-i7\pi/6} = 4e^{i5\pi/6}$

II.3 $-2 + 3i$; a quarter turn about 0

II.4 $4e^{i\pi/3} = 2 + 2\sqrt{3}\,i$

II.5 $2 + 2i = 2\sqrt{2}\,e^{i\pi/4}$; no: $2\sqrt{2} \neq 4$

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