Polar and Euler forms

Skills sheet · HL AA · AHL 1.13

KEY FACTS

$z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{\mathrm{i}\theta}$ D1

θ from arctan $\left|\frac{b}{a}\right|$ and the quadrant P1

$r_1\mathrm{e}^{\mathrm{i}\theta_1}\, r_2\mathrm{e}^{\mathrm{i}\theta_2} = r_1 r_2\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}$ P2

$\times r\mathrm{e}^{\mathrm{i}\alpha}$: rotation by α, enlargement by r P3

One number, two forms: exact values P4

arctan $\frac{b}{a}$ alone fails in two quadrants W1.

$r > 0$ and $-\pi < \theta \leq \pi$ D1.

A sum: back to Cartesian W2.

WHAT UNLOCKS

- "modulus and argument" $\rightarrow$ a sketch, $r = \sqrt{a^2+b^2}$, then the quadrant.
- a product, a quotient $\rightarrow$ Euler form: multiply the moduli, add the arguments.
- "rotated about O", "enlarged" $\rightarrow$ multiply by $r\mathrm{e}^{\mathrm{i}\alpha}$.
- "hence the exact value of cos" $\rightarrow$ the same product in both forms, then equate parts.

WORKED EXAMPLE ①

Write $3-3\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. no GDC

$r = \sqrt{9+9} = 3\sqrt{2}$; fourth quadrant, $\arctan 1 = \frac{\pi}{4}$ (M1)

$3\sqrt{2}\,\mathrm{e}^{-\mathrm{i}\pi/4}$ A1

Method

Cartesian to Euler

① Sketch: which quadrant?
② r, then the reference angle.
③ Adjust into $(-\pi, \pi]$.

WORKED EXAMPLE ②

Write $4\mathrm{e}^{\mathrm{i}2\pi/3}$ in the form $a+b\mathrm{i}$. no GDC

$4\cos\frac{2\pi}{3} + 4\mathrm{i}\sin\frac{2\pi}{3} = 4\left(-\frac{1}{2}\right) + 4\cdot\frac{\sqrt{3}}{2}\,\mathrm{i}$ (M1)

$-2+2\sqrt{3}\,\mathrm{i}$ A1

Method

Euler to Cartesian

① $a = r\cos\theta$, $b = r\sin\theta$.
② Exact values of cos, sin.
③ Simplify.

FLUENCY

1. -5 in Euler form no GDC

2. $3\mathrm{i}$ in Euler form no GDC

3. $2\mathrm{e}^{\mathrm{i}\pi/3} \times 3\mathrm{e}^{\mathrm{i}\pi/6}$ no GDC

4. $\dfrac{8\mathrm{e}^{\mathrm{i}\pi/2}}{2\mathrm{e}^{-\mathrm{i}\pi/4}}$ no GDC

5. $\arg(-2-2\mathrm{i})$ no GDC

6. $\mathrm{e}^{\mathrm{i}\pi}+1$ no GDC

REASONING

●●● **Question 1.** Let $z = 1 - \sqrt{3}\,\mathrm{i}$. [no GDC]

(a) Find the modulus and the argument of z. [3]

●●● **Question 2.** Let $z = 2\mathrm{e}^{\mathrm{i}\pi/3}$ and $w = \sqrt{2}\,\mathrm{e}^{\mathrm{i}3\pi/4}$. [no GDC]

(a) Find zw and $\dfrac{z}{w}$, in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [4]

●●● **Question 3.** The point A represents $3 + \mathrm{i}$ in the complex plane. It is rotated anticlockwise by $\frac{\pi}{2}$ about O, then enlarged with scale factor 2, centre O, to give the point B. [no GDC]

(a) Find the complex number represented by B. [3]

PROBLEMS

●●● **Question 4.** Let $u = \sqrt{3} - \mathrm{i}$ and $v = 1 + \mathrm{i}$. [no GDC]

(a) Find uv in the form $a + b\mathrm{i}$. [2]

(b) Write u and v in the form $r\mathrm{e}^{\mathrm{i}\theta}$. Hence write uv in the same form. [3]

(c) Hence find the exact value of $\sin\dfrac{\pi}{12}$. [2]

●●● **Question 5.** The points O, A and B form an equilateral triangle. A represents $4 + 2\mathrm{i}$, and B is the image of A by the anticlockwise rotation of angle $\frac{\pi}{3}$ about O. [no GDC]

(a) Find the complex number represented by B, in exact Cartesian form. [4]

(b) Find the exact area of triangle OAB. [2]

ANSWERS

1 $5e^{i\pi}$

2 $3e^{i\pi/2}$

3 $6e^{i\pi/2} = 6i$

4 $4e^{i3\pi/4}$

5 $-\frac{3\pi}{4}$ (third quadrant)

6 0

Q1 $|z| = 2$; fourth quadrant, $\arctan\sqrt{3} = \frac{\pi}{3}$: $\arg z = -\frac{\pi}{3}$

Q2 $zw = 2\sqrt{2}\,e^{i13\pi/12} = 2\sqrt{2}\,e^{-i11\pi/12}$; $\frac{z}{w} = \sqrt{2}\,e^{-i5\pi/12}$

Q3 multiply by $2e^{i\pi/2} = 2i$: $2i(3+i) = -2+6i$

Q4 $(\sqrt{3}+1)+(\sqrt{3}-1)i$; $u = 2e^{-i\pi/6}$, $v = \sqrt{2}\,e^{i\pi/4}$, $uv = 2\sqrt{2}\,e^{i\pi/12}$; $2\sqrt{2}\sin\frac{\pi}{12} = \sqrt{3}-1$, $\sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$

Q5 $(4+2i)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = (2-\sqrt{3})+(1+2\sqrt{3})i$; area $\frac{\sqrt{3}}{4}\times 20 = 5\sqrt{3}$

Written and verified by a practising IB mathematics teacher, produced with AI assistance; every result recomputed by software, every figure computed from its function. Not affiliated with or endorsed by the International Baccalaureate Organization. © 2026 Hence