Proof by induction and contradiction

HL AA · Proof and complex numbers · AHL 1.15

Checking a statement for $n = 1, 2, 3, \ldots, 100$ proves nothing about $n = 101$. Induction proves it for every positive integer with two facts: it is true at the start, and whenever it is true for one integer it is true for the next. Contradiction proves a statement by showing that its opposite is impossible; a single counterexample, explained, is enough to show that a statement is false.

I PROOF BY MATHEMATICAL INDUCTION

AHL 1.15 · basis, assumption, inductive step, conclusion

I.1 The four parts, on a sum

Definition 1 *the principle of induction*

Let $P(n)$ be a statement about the positive integer n. If **(1)** $P(1)$ is true, and **(2)** for every $k \in \mathbb{Z}^+$, $P(k)$ true implies $P(k+1)$ true, then $P(n)$ is true for every $n \in \mathbb{Z}^+$. The start can be another integer, for example $n \geq 2$.

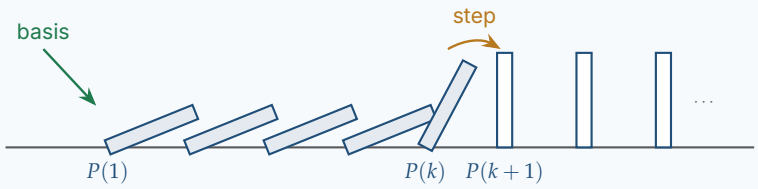


Figure 1 — The dominoes of induction: the first one falls (the basis), and each one that falls knocks over the next (the inductive step); so every one falls.

Property 1 *the four parts the markscheme looks for*

Basis: show $P(1)$, computing both sides. **Assumption**: "assume $P(k)$ is true for some $k \in \mathbb{Z}^+$", written out. **Step**: start from the expression for $k+1$ and use the assumption to reach the statement for $k+1$. **Conclusion**: "$P(1)$ is true, and $P(k)$ true implies $P(k+1)$ true; so $P(n)$ is true for all $n \in \mathbb{Z}^+$." The conclusion earns its own R1 mark only if the first marks were earned.

Example 1

Prove by induction that $\sum_{r=1}^{n}(3r-1) = \dfrac{n(3n+1)}{2}$ for all $n \in \mathbb{Z}^+$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Basis: $n = 1$, both sides	left $3-1=2$, right $\frac{1\times4}{2}=2$: true
Step 2 Assumption, for some k	$\sum_{r=1}^{k}(3r-1) = \dfrac{k(3k+1)}{2}$ for some $k \in \mathbb{Z}^+$
Step 3 Step: add the $(k+1)$th term	$\sum_{r=1}^{k+1}(3r-1) = \dfrac{k(3k+1)}{2} + (3k+2)$
Step 4 Simplify	$= \dfrac{3k^2+7k+4}{2} = \dfrac{(k+1)(3k+4)}{2}$
Step 5 Reach the statement for $k+1$	$= \dfrac{(k+1)\,(3(k+1)+1)}{2}$
Step 6 Conclusion	true for $n=1$, and true for $n=k$ implies true for $n=k+1$; so true for all $n \in \mathbb{Z}^+$

Watch out 1

"Let $n = k$ be true" or "assume true for all k" loses the assumption mark: the assumption is for *some* k. Never start the step from the result for $k+1$ that you want to prove.

APPLY

1 The basis for $\sum_{r=1}^{n} 2^r = 2^{n+1} - 2$.

2 The term added in the step for $\sum_{r=1}^{n} r^3$.

I.2 Divisibility

Property 2 *the four kinds of statement*

Sums: $\sum_{r=1}^{k+1} = \sum_{r=1}^{k} + (\text{term } k+1)$. **Divisibility**: write $f(k+1)$ as a multiple of $f(k)$ plus a multiple of the divisor. **Derivatives**: differentiate the formula for the kth derivative. **Inequalities**: from the assumption, a chain of inequalities to the statement for $k+1$.

Example 2

Prove by induction that $6^n - 1$ is divisible by 5 for all $n \in \mathbb{Z}^+$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Basis: $n = 1$	$6-1=5$: true
Step 2 Assumption: $f(k) = 5m$	assume $6^k - 1 = 5m$, $m \in \mathbb{Z}$, for some $k \in \mathbb{Z}^+$
Step 3 Step: bring in the assumption	$6^{k+1} - 1 = 6 \times 6^k - 1 = 6(5m+1) - 1 = 30m + 5$
Step 4 A multiple of 5	$= 5(6m+1)$
Step 5 Conclusion	true for $n=1$, and true for $n=k$ implies true for $n=k+1$; so true for all $n \in \mathbb{Z}^+$

APPLY

3 The step for "$11^n - 1$ is divisible by 10".

4 The step for "$n^3 - n$ is divisible by 3".

I.3 Derivatives and recurrences

Example 3

$f(x) = xe^{-x}$. Prove by induction that $f^{(n)}(x) = (-1)^n(x-n)e^{-x}$ for all $n \in \mathbb{Z}^+$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Basis: $n = 1$	$f'(x) = e^{-x} - xe^{-x} = (-1)(x-1)e^{-x}$: true
Step 2 Assumption	assume $f^{(k)}(x) = (-1)^k(x-k)e^{-x}$ for some $k \in \mathbb{Z}^+$
Step 3 Step: differentiate the kth derivative	$f^{(k+1)}(x) = (-1)^k\left(e^{-x} - (x-k)e^{-x}\right)$
Step 4 Factor e^{-x}	$= (-1)^k(1+k-x)e^{-x}$
Step 5 Reach the statement for $k+1$	$= (-1)^{k+1}\left(x-(k+1)\right)e^{-x}$
Step 6 Conclusion	true for $n = 1$, and true for $n = k$ implies true for $n = k+1$; so true for all $n \in \mathbb{Z}^+$

APPLY

5 $u_1 = 1$, $u_{n+1} = u_n + 2n + 1$: the step that proves $u_n = n^2$.

II PROOF BY CONTRADICTION, AND COUNTEREXAMPLES

AHL 1.15 · assume the opposite; one explained counterexample

II.1 Proof by contradiction

Definition 2 *proof by contradiction*

To prove a statement S, assume that S is **false** and deduce something impossible (an even number equal to an odd number, a fraction in lowest terms that is not). The assumption was wrong, so S is true.

Property 3 *irrational numbers*

To prove that a number is irrational, assume it is rational, $\frac{p}{q}$ with $p, q \in \mathbb{Z}, q \neq 0$ (in lowest terms if needed), and reach a contradiction. For $\log_2 5 = \frac{p}{q}$ with $p, q > 0$: $2^{p/q} = 5$, so $2^p = 5^q$ — even equal to odd, impossible.

Example 4

Prove by contradiction that $\sqrt{2}$ is irrational.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Assume the opposite	$\sqrt{2} = \frac{p}{q}$, $p, q \in \mathbb{Z}^+$, in lowest terms
Step 2 Square	$p^2 = 2q^2$: p^2 is even, so p is even, $p = 2m$
Step 3 Substitute	$4m^2 = 2q^2$, $q^2 = 2m^2$: q is even too
Step 4 Contradiction	p and q both even: not in lowest terms. So $\sqrt{2}$ is irrational

Watch out 2

In a proof by contradiction, state the assumption clearly at the start ("Assume that $\log_2 5$ is rational") and say clearly where the contradiction is. A counterexample stated without the reason earns no mark.

APPLY

1 Prove that $\log_3 10$ is irrational: the contradiction.

II.2 Counterexamples

Property 4 *counterexamples*

One case is enough to show that a statement "for all n" is **false**, but it must be **explained**: give the value and show why the statement fails there. Thousands of cases that work never prove it true.

Example 5

Show that the statement "$2^n + 3$ is prime for every $n \in \mathbb{Z}^+$" is false.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Try values	$5,\ 7,\ 11,\ 19$ for $n = 1$ to 4: all prime
Step 2 The counterexample	$n = 5$: $2^5 + 3 = 35$
Step 3 Why it fails	$35 = 5 \times 7$ is not prime

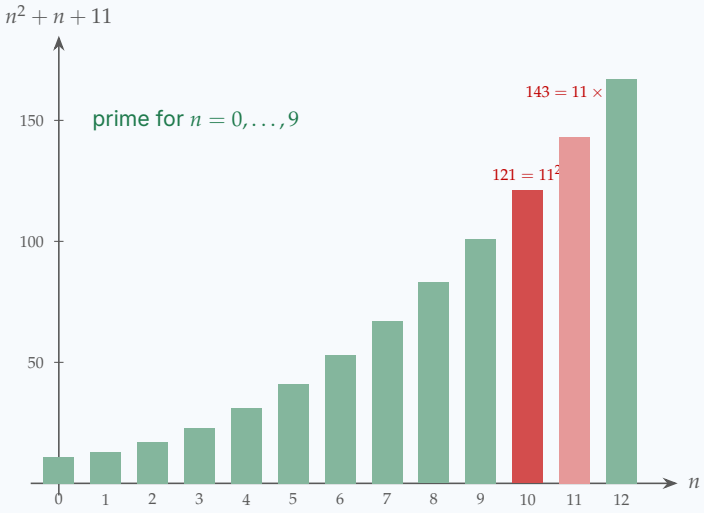


Figure 2 — Exam-style question 4 (b): $n^2 + n + 11$ for $n = 0$ to 12. Prime for $n = 0$ to 9, then $121 = 11^2$ at $n = 10$ — one counterexample, and the claim "always prime" is false.

APPLY

2 "$n^2 - n + 41$ is always prime".

3 "$x^2 > x$ for every real x".

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A Induction: a sum

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 7]

Prove by mathematical induction that

$$\sum_{r=1}^{n} r \times 2^r = (n-1)2^{n+1} + 2 \quad \text{for all } n \in \mathbb{Z}^+.$$

WORKING

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Basis: $n = 1$	$n = 1$: $1 \times 2 = 2$ and $0 \times 4 + 2 = 2$, true	A1
Step 2 Assumption, for some k	assume true for $n = k$: $\sum_{r=1}^{k} r2^r = (k-1)2^{k+1} + 2$ *For some $k \in \mathbb{Z}^+$.*	M1
Step 3 Add the $(k+1)$th term	$\sum_{r=1}^{k+1} r2^r = (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$ *Adding the $(k+1)$th term.*	M1
Step 4 Factor 2^{k+1}	$= ((k-1) + (k+1))\,2^{k+1} + 2$	A1
Step 5 Simplify	$= 2k \times 2^{k+1} + 2$	A1
Step 6 Reach the statement for $k+1$	$= k \times 2^{k+2} + 2 = ((k+1) - 1)\,2^{(k+1)+1} + 2$	A1
Step 7 Conclusion	true for $n = 1$, and true for $n = k$ implies true for $n = k+1$; hence true for all $n \in \mathbb{Z}^+$ *Only if the three method or accuracy marks before were earned.*	R1

B Induction: divisibility

EXAM-STYLE QUESTION 2 — Paper 1 · no calculator · [Maximum mark: 7]

Prove by mathematical induction that $3^{2n+1} + 2^{n+2}$ is divisible by 7 for all $n \in \mathbb{Z}^+$.

WORKING

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Basis: $n = 1$	$n = 1$: $27 + 8 = 35 = 7 \times 5$	A1
Step 2 Assumption: $f(k) = 7m$	assume $3^{2k+1} + 2^{k+2} = 7m$, $m \in \mathbb{Z}$, for some k	M1
Step 3 The expression for $k+1$	consider $3^{2(k+1)+1} + 2^{(k+1)+2} = 3^{2k+3} + 2^{k+3}$	M1
Step 4 Split the powers	$= 9 \times 3^{2k+1} + 2 \times 2^{k+2}$	A1
Step 5 Bring in the assumption	$= 9\left(3^{2k+1} + 2^{k+2}\right) - 7 \times 2^{k+2}$	A1
Step 6 A multiple of 7	$= 7\left(9m - 2^{k+2}\right)$, a multiple of 7	A1
Step 7 Conclusion	true for $n = 1$, and true for $n = k$ implies true for $n = k+1$; hence true for all $n \in \mathbb{Z}^+$	R1

C Induction: derivatives

EXAM-STYLE QUESTION 3 — Paper 1 · no calculator · [Maximum mark: 8]

Let $f(x) = xe^{2x}$.

(a) Find $f'(x)$ and $f''(x)$. [2]

(b) Prove by mathematical induction that $f^{(n)}(x) = 2^{n-1}(2x + n)e^{2x}$ for all $n \in \mathbb{Z}^+$. [6]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Product rule	$f'(x) = (2x+1)e^{2x}$	A1
Step 2 Second derivative	$f''(x) = (4x+4)e^{2x}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Basis: $n = 1$	$n = 1$: $2^0(2x+1)e^{2x} = f'(x)$, true	A1
Step 2 Assumption	assume $f^{(k)}(x) = 2^{k-1}(2x+k)e^{2x}$ for some k	M1
Step 3 Differentiate the kth derivative	$f^{(k+1)}(x) = 2^{k-1}\left(2e^{2x} + 2(2x+k)e^{2x}\right)$	M1A1
	M1 for differentiating the assumption with the product rule.	
Step 4 Reach the statement for $k+1$	$= 2^k(2x+k+1)e^{2x}$	A1
Step 5 Conclusion	true for $n = 1$, and true for $n = k$ implies true for $n = k+1$; hence true for all $n \in \mathbb{Z}^+$	R1

D Contradiction and counterexamples

EXAM-STYLE QUESTION 4 Paper 1 · no calculator · [Maximum mark: 6]

Answer the following without a calculator.

(a) Prove by contradiction that $\log_2 5$ is irrational. [4]

(b) Show that the statement "$n^2 + n + 11$ is prime for every $n \in \mathbb{N}$" is false. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Assume the opposite	assume $\log_2 5 = \frac{p}{q},\ p, q \in \mathbb{Z}^+$ $\log_2 5 > 0$, *so* $p, q > 0$.	M1
Step 2 Remove the logarithm	$2^{p/q} = 5,\ 2^p = 5^q$	A1
Step 3 Even and odd	2^p is even and 5^q is odd	R1
Step 4 Contradiction	contradiction, so $\log_2 5$ is irrational	R1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The counterexample	$n = 10$: $100 + 10 + 11 = 121$	A1
Step 2 Why it fails	$121 = 11 \times 11$ is not prime	R1

E A conjecture on Paper 2

EXAM-STYLE QUESTION 5 Paper 2 · calculator · [Maximum mark: 7]

A sequence is defined by $u_1 = 3$ and $u_{n+1} = 2u_n - 1$ for $n \in \mathbb{Z}^+$.

(a) Find u_2, u_3, u_4 and u_5. [1]

(b) Suggest an expression for u_n in terms of n. [1]

(c) Prove your answer to part (b) by mathematical induction. [5]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Use the recurrence	$5,\ 9,\ 17,\ 33$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Read the pattern	$u_n = 2^n + 1$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Basis: $n = 1$	$n = 1$: $2^1 + 1 = 3 = u_1$	A1
Step 2 Assumption	assume $u_k = 2^k + 1$ for some $k \in \mathbb{Z}^+$	M1
Step 3 Use the recurrence	$u_{k+1} = 2u_k - 1 = 2\left(2^k + 1\right) - 1$	M1
Step 4 Simplify	$= 2^{k+1} + 1$	A1
Step 5 Conclusion	true for $n = 1$, and true for $n = k$ implies true for $n = k + 1$; hence true for all $n \in \mathbb{Z}^+$	R1

IV WHICH METHOD? READ THE QUESTION

AHL 1.15 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"prove by induction"	basis, assumption, step, conclusion	all four, in words P1
a sum $\sum_{r=1}^{n}$	add the $(k+1)$th term	to the assumed sum P2
"divisible by d"	$f(k) = dm$	$f(k+1) = a\,f(k) + d \times (\ldots)$ P2
"$\frac{\mathrm{d}^n}{\mathrm{d}x^n}$"	differentiate the kth derivative	factor to the formula for $k+1$ P2
"prove … is irrational"	assume $= \frac{p}{q}$	reach an impossibility, then say "contradiction" P3
"show that … is not always true"	one counterexample	the value and why it fails P4

In the exam 1 *command terms*

Prove: every step justified; the conclusion written. **Show that the statement is false**: a counterexample with its explanation. **Hence true for all** n: R1, given only after the earlier marks.

V ON THE GDC: PAPER 2

AHL 1.15 · NumWorks · every calculator move of this topic

1 Testing a conjecture on the GDC

Sequences → Add a sequence → Recursive first order → Table

A table gives the first terms to guess a formula; it proves nothing. **On paper:** the conjecture, then the proof by induction.

SEQUENCES

```
u(n+1)=2u(n)-1
u(1)=3
n u(n)
1 3
2 5
3 9
4 17
5 33
```

ANSWERS TO THE APPLY QUESTIONS

I.1 $n = 1$: $2 = 4 - 2$

I.2 $(k+1)^3$

I.3 $11^{k+1} - 1 = 11\left(11^k - 1\right) + 10$

I.4 $(k+1)^3 - (k+1) = \left(k^3 - k\right) + 3\left(k^2 + k\right)$

I.5 $u_{k+1} = k^2 + 2k + 1 = (k+1)^2$

II.1 $3^p = 10^q$: odd equal to even

II.2 $n = 41$: $41^2 = 41 \times 41$

II.3 $x = \frac{1}{2}$: $\frac{1}{4} < \frac{1}{2}$

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