Proof by induction and contradiction

Skills sheet · HL AA · AHL 1.15

KEY FACTS

$P(1)$, and $P(k) \Rightarrow P(k+1)$: $P(n)$ for all n D1

Basis, assumption, step, conclusion P1

Sum, divisibility, derivative, inequality P2

Contradiction: assume false, reach the impossible D2

One counterexample, **explained** P4

Assume for some k W1.

Never start from the $k+1$ result W1.

Say where the contradiction is W2.

WHAT UNLOCKS

- $\sum_{r=1}^{n} \rightarrow$ the assumed sum plus the $(k+1)$th term, then factorise.
- "divisible by d" $\rightarrow$ $f(k) = dm$; write $f(k+1) = a\,f(k) + d \times (\ldots)$.
- "$f^{(n)}(x) = \ldots$" $\rightarrow$ differentiate the formula for $f^{(k)}$.
- "irrational", "no integers such that" $\rightarrow$ assume the opposite, find the impossibility.

WORKED EXAMPLE ①

Assuming $\sum_{r=1}^{k}(2r-1) = k^2$, show that $\sum_{r=1}^{k+1}(2r-1) = (k+1)^2$. no GDC

$\sum_{r=1}^{k+1}(2r-1) = k^2 + (2(k+1)-1)$ M1

$= k^2 + 2k + 1 = (k+1)^2$ A1

Method

The step for a sum

① The assumed sum.
② Plus the term $r = k+1$.
③ Factorise to the target.

WORKED EXAMPLE ②

Show that "$2^n + 1$ is prime for every $n \in \mathbb{Z}^+$" is false. no GDC

$n = 3$: $2^3 + 1 = 9$ A1

$9 = 3 \times 3$ is not prime R1

Method

A counterexample

① Try small values.
② Give the value of n.
③ Say why it fails.

FLUENCY

●○○ **1.** Check $n = 1$ in $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ no GDC

●○○ **2.** The $(k+1)$th term of $\sum_{r=1}^{n} r(r+1)$ no GDC

●○○ **3.** $5^{k+1} - 1$ in terms of $5^k - 1$ no GDC

●○○ **4.** $\dfrac{\mathrm{d}}{\mathrm{d}x}\left((x+k)\mathrm{e}^x\right)$ no GDC

●○○ **5.** A counterexample to "$n^2 \geq 2n+1$ for all $n \in \mathbb{Z}^+$" no GDC

●○○ **6.** The assumption that starts a proof that $\sqrt{2}$ is irrational no GDC

REASONING

●●● **Question 1.** Prove by mathematical induction that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{Z}^+$. [no GDC]

(a) Write the four parts of the proof. [6]

●●● **Question 2.** Prove by mathematical induction that $7^n - 3^n$ is divisible by 4 for all $n \in \mathbb{Z}^+$. [no GDC]

(a) Write the four parts of the proof. [6]

●●● **Question 3.** Prove by contradiction that there are no integers a and b such that $15a + 6b = 1$. [no GDC]

(a) Write the proof. [3]

PROBLEMS

●●● **Question 4.** Let $f(x) = \frac{1}{1+x}$, $x > -1$. [no GDC]

(a) Find $f'(x)$ and $f''(x)$. [2]

(b) Prove by mathematical induction that $f^{(n)}(x) = \frac{(-1)^n n!}{(1+x)^{n+1}}$ for all $n \in \mathbb{Z}^+$. [5]

●●● **Question 5.** Answer the following without a calculator. [no GDC]

(a) Prove by mathematical induction that $3^n > n^2 + n$ for all $n \in \mathbb{Z}^+$. [6]

(b) Show that the statement "$3^n > n^3$ for all $n \in \mathbb{Z}^+$" is false. [2]

ANSWERS

1 both sides equal 1

2 $(k+1)(k+2)$

3 $5\left(5^k-1\right)+4$

4 $(x+k+1)\mathrm{e}^x$

5 $n=1$: $1<3$ (or $n=2$: $4<5$)

6 $\sqrt{2}=\frac{p}{q}$, $p,q\in\mathbb{Z}^+$, in lowest terms

Q1 $n=1$: $\frac{1}{2}=\frac{1}{2}$; $\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{k+1}{k+2}$; conclusion

Q2 $n=1$: 4; $7^{k+1}-3^{k+1}=7\left(7^k-3^k\right)+4\times 3^k=4\left(7m+3^k\right)$; conclusion

Q3 assume such a,b exist: $15a+6b=3(5a+2b)$ is a multiple of 3, but 1 is not: contradiction

Q4 $f'(x)=-\frac{1}{(1+x)^2}$, $f''(x)=\frac{2}{(1+x)^3}$; step: $\frac{\mathrm{d}}{\mathrm{d}x}\left((-1)^k k!(1+x)^{-(k+1)}\right)=(-1)^{k+1}(k+1)!(1+x)^{-(k+2)}$

Q5 $n=1$: $3>2$; $3^{k+1}>3k^2+3k\geq k^2+3k+2=(k+1)^2+(k+1)$ since $2k^2-2\geq 0$; (b) $n=3$: $27=27$, not greater

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