Equations of planes

A plane is fixed by one point and one direction perpendicular to it, its normal. From that comes the most useful equation of the topic, $ax + by + cz = d$, whose coefficients are the normal. A plane can also be swept out from a point by two directions, which is how three points give a plane: two vectors between them, their vector product as the normal, and one substitution for d.

I THREE FORMS OF THE EQUATION OF A PLANE

AHL 3.17 · $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, $ax + by + cz = d$

I.1 The normal form and the Cartesian equation

Definition 1 *the normal form*

A **normal n** to a plane is a vector perpendicular to every line in it. The plane through A with normal $\mathbf{n}$ is

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n},$$

and with $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ this is the **Cartesian equation**

$$ax + by + cz = d.$$

The coefficients of x, y, z **are** the normal.

Example 1

Find the Cartesian equation of the plane through $A(2, -1, 3)$ with normal $\mathbf{n} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 $d = \mathbf{a} \cdot \mathbf{n}$	$2 - 4 - 6 = -8$
Step 2 The coefficients are the normal	$x + 4y - 2z = -8$

APPLY

1 A normal to $3x - z = 7$.

2 The plane through $(0, 2, -1)$ with normal $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

I.2 The parametric form

Definition 2 *the parametric form*

The plane through A containing the non-parallel directions $\mathbf{b}$ and $\mathbf{c}$ is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}, \qquad \lambda, \mu \in \mathbb{R}.$$

Its normal is $\mathbf{n} = \mathbf{b} \times \mathbf{c}$.

Example 2

Find the Cartesian equation of the plane $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 $\mathbf{n} = \mathbf{b} \times \mathbf{c}$	$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$
Step 2 d from the point	$1 - 0 + 4 = 5$
Step 3 The equation	$x - y + 2z = 5$
Check with $\lambda = 1$, $\mu = -1$: the point $(2, -1, 1)$ gives $2 + 1 + 2 = 5$.	

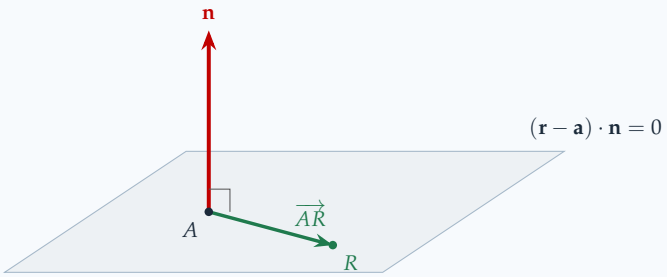


Figure 1 — Every point R of the plane gives $\overrightarrow{AR}$ perpendicular to $\mathbf{n}$: $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$.

APPLY

3 A normal to $\mathbf{r} = \mathbf{a} + \lambda \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$.

I.3 A plane through three points

Property 1 *a plane through three points*

For A, B, C not on one line: $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$, then $d = \mathbf{a} \cdot \mathbf{n}$. Check the equation with B and C: they must satisfy it too.

Example 3

Find the Cartesian equation of the plane through $A(1, 1, 0)$, $B(3, 2, 1)$ and $C(0, 3, 2)$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Two vectors from A	$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$
Step 2 The normal: their product	$\begin{pmatrix} 0 \\ -5 \\ 5 \end{pmatrix}$, or $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$
Step 3 d from A	$y - z = 1$
Step 4 Check with B and C	$2 - 1 = 1$, $3 - 2 = 1$

Watch out 1

d is $\mathbf{a} \cdot \mathbf{n}$, a number found from a point of the plane, never the length of $\mathbf{n}$ or a coordinate read off.

APPLY

4 The plane through $(2, 0, 0)$, $(0, 4, 0)$ and $(0, 0, 1)$.

II READING A PLANE: NORMALS, POINTS, PARALLEL PLANES

AHL 3.17 · parallel planes, a line perpendicular to a plane, from one form to another

II.1 Parallel planes and perpendicular lines

Property 2 *parallel planes, perpendicular lines*

Planes are **parallel** when their normals are parallel: $2x - y + 2z = 9$ and $2x - y + 2z = 3$ share the normal $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$. A line **perpendicular** to a plane has the normal as its direction: $\mathbf{r} = \mathbf{p} + t\mathbf{n}$.

Example 4

$\Pi: \ 3x - 2y + z = 4$. Find the plane parallel to Π through $P(1, -1, 2)$, and the line through P perpendicular to Π.

WHAT YOU DO	WHAT YOU WRITE
Step 1 The same normal	$\mathbf{n} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$
Step 2 A new d from P	$3 + 2 + 2 = 7$: $3x - 2y + z = 7$
Step 3 The line: direction $\mathbf{n}$	$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

Watch out 2

A point of the plane must satisfy its equation: check it before using it. A direction in the plane must be perpendicular to $\mathbf{n}$: check $\mathbf{b} \cdot \mathbf{n} = 0$.

APPLY

1 Is $(2, 1, 0)$ in $3x - 2y + z = 4$?

2 Are $x + 2y - z = 3$ and $-2x - 4y + 2z = 1$ parallel?

II.2 From Cartesian to parametric

Property 3 *from Cartesian to parametric*

To write $ax + by + cz = d$ as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$: find one point of the plane, then two non-parallel directions $\mathbf{b}$, $\mathbf{c}$ with $\mathbf{b} \cdot \mathbf{n} = \mathbf{c} \cdot \mathbf{n} = 0$.

Example 5

Write the plane $x + 3y - z = 6$ in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 A point, checked	$(6, 0, 0)$: $6 = 6$
Step 2 Two directions with $\mathbf{b} \cdot \mathbf{n} = 0$	$\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, not parallel
Step 3 The parametric form	$\mathbf{r} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

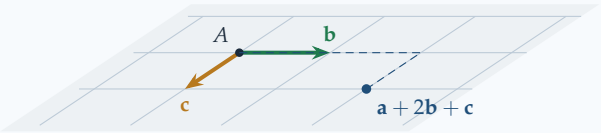


Figure 2 — From A, the steps $\lambda\mathbf{b} + \mu\mathbf{c}$ reach every point of the plane: a grid drawn with the two directions.

APPLY

3 $4x + y + 2z = 8$ in parametric form.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A A plane through three points

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 6]

The points $A(1, 0, 2)$, $B(2, 1, 0)$ and $C(0, 3, 1)$ lie in the plane Π.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [2]

(b) Hence find a vector normal to Π. [2]

(c) Find the Cartesian equation of Π. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $\overrightarrow{AB}$	$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$	A1
Step 2 $\overrightarrow{AC}$	$\overrightarrow{AC} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The vector product	$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -1+6 \\ 2+1 \\ 3+1 \end{pmatrix}$	M1
Step 2 The normal	$\mathbf{n} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$ *Or any non-zero multiple.*	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $d = \mathbf{a} \cdot \mathbf{n}$	$5 \times 1 + 3 \times 0 + 4 \times 2 = 13$ *Substitute a point.*	M1
Step 2 The equation	$5x + 3y + 4z = 13$	A1

B Reading a plane

EXAM-STYLE QUESTION 2 — Paper 1 · no calculator · [Maximum mark: 6]

The plane Π has equation $2x - y + 2z = 9$.

(a) Write down a vector normal to Π. [1]

(b) Find the Cartesian equation of the plane parallel to Π that passes through $(1, 1, 1)$. [2]

(c) Write the equation of Π in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Read the coefficients	$\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Same normal, d from the point	$2 - 1 + 2 = 3$	M1
Step 2 The equation	$2x - y + 2z = 3$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 A point of the plane, checked	a point, for example $(4, 1, 1)$: $8 - 1 + 2 = 9$	A1
	Any point of Π.	
Step 2 Two directions with $\mathbf{b} \cdot \mathbf{n} = 0$	$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, as $\mathbf{b} \cdot \mathbf{n} = 0$	A1
	Any direction perpendicular to $\mathbf{n}$.	
Step 3 The parametric form	$\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$	A1
	Two non-parallel directions.	

C On the GDC (Paper 2)

EXAM-STYLE QUESTION 3 Paper 2 · calculator · [Maximum mark: 6]

The plane Π contains the points $A(2, 1, -1)$, $B(4.5, 0, 3)$ and $C(-1, 4, 2)$.

(a) Find a vector normal to Π. [2]

(b) Find the Cartesian equation of Π, with integer coefficients. [2]

(c) The point $D(k, 2, 0)$ lies in Π. Find the value of k. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The vector product	$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2.5 \\ -1 \\ 4 \end{pmatrix} \times \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix}$	M1
Step 2 Compute	$\begin{pmatrix} -15 \\ -19.5 \\ 4.5 \end{pmatrix}$, or $\begin{pmatrix} 10 \\ 13 \\ -3 \end{pmatrix}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 d from A	$10 \times 2 + 13 \times 1 - 3 \times (-1) = 36$	M1
Step 2 The equation	$10x + 13y - 3z = 36$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Substitute D	$10k + 26 = 36$	M1
Step 2 Solve	$k = 1$	A1

IV WHICH METHOD? READ THE QUESTION

AHL 3.17 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
$ax + by + cz = d$	the normal is (a, b, c)	read it off, or any multiple D1
a point and a normal	$d = \mathbf{a} \cdot \mathbf{n}$	$ax + by + cz = d$ with that d D1
three points A, B, C	$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$	then d from A, checked with B and C P1
$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$	$\mathbf{n} = \mathbf{b} \times \mathbf{c}$	the Cartesian equation from $\mathbf{n}$ and A D2
"the plane parallel to Π through P"	the same normal	a new d from P P2
"the line through P perpendicular to Π"	direction $\mathbf{n}$	$\mathbf{r} = \mathbf{p} + t\mathbf{n}$ P2
"write in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$"	a point and two directions with $\mathbf{b} \cdot \mathbf{n} = 0$	check each in the equation P3
"does D lie in the plane?"	its coordinates in the equation	equal to d or not W2

In the exam 1 *command terms*

Write down: no working needed, the normal read from the equation. **Find the Cartesian equation**: in the form $ax + by + cz = d$, integers when possible. **Hence**: use the normal just found.

V ON THE GDC: PAPER 2

AHL 3.17 · NumWorks · every calculator move of this topic

1 A normal from three points

Calculation → cross([[2.5][-1][4]],[[-3][3][3]])

The vector product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$; divide by a common factor to get small whole numbers. **On paper:** the two vectors, then the normal.

CALCULATIONS

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[[-15][-19.5][4.5]]
```

2 The value of d

Calculation → dot([[10][13][-3]],[[2][1][-1]])

$d = \mathbf{n} \cdot \mathbf{a}$ with any point of the plane. **On paper:** $10x + 13y - 3z = 36$.

CALCULATIONS

```
dot(…)=36
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ANSWERS TO THE APPLY QUESTIONS

I.1 $\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$

I.2 $2x + y + z = 1$

I.3 $\begin{pmatrix} -1 \\ -6 \\ 2 \end{pmatrix}$

I.4 $2x + y + 4z = 4$

II.1 yes: $6 - 2 + 0 = 4$

II.2 yes: the normals are multiples (-2); distinct planes

II.3 for example $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

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