Intersections, angles and systems

HL AA · Vectors, lines and planes · AHL 3.18 · AHL 1.16

Where a line pierces a plane, where two planes cut each other, whether three planes share a point, a line or nothing: each question is a system of equations, and its answer is a picture. The angles come from the normals and the directions you already have. This is the part of the topic that Paper 1 and Paper 2 like to finish on.

I A LINE AND A PLANE, TWO PLANES

AHL 3.18 · a line meeting a plane, the line of two planes, angles with $\mathbf{n}$ and $\mathbf{b}$

I.1 A line and a plane

Property 1 *a line and a plane*

Substitute the parametric form of the line into the equation of the plane: one equation in λ.
One solution: the line meets the plane at one point.
No solution ($\mathbf{b}\cdot\mathbf{n}=0$, a false equation): the line is parallel to the plane.
Every λ ($\mathbf{b}\cdot\mathbf{n}=0$, a true equation): the line lies in the plane.

Example 1

Find where the line $\mathbf{r}=\begin{pmatrix}2\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}$ meets the plane $2x+y-z=9$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Substitute the line	$2(2+\lambda)+(-\lambda)-(1+3\lambda)=9$
Step 2 Collect	$3-2\lambda=9$
Step 3 Solve	$\lambda=-3$
Step 4 The point	$(-1,\,3,\,-8)$

APPLY

1 $\mathbf{r}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\0\end{pmatrix}$ and $x+y+z=5$.

2 $\mathbf{r}=\begin{pmatrix}0\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $3x-y+z=6$.

I.2 Angles

Property 2 *angles*

Between a line and a plane: $\sin\theta = \dfrac{|\mathbf{b}\cdot\mathbf{n}|}{|\mathbf{b}|\,|\mathbf{n}|}$, **sine**, because $\mathbf{n}$ is perpendicular to the plane. Between two planes: the angle between their normals, $\cos\theta = \dfrac{|\mathbf{n}_1\cdot\mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|}$.

Example 2

Find the acute angle between the line of Example 1 and the plane $2x + y - z = 9$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 $\mathbf{b}\cdot\mathbf{n}$, with the modulus	$\lvert 2 - 1 - 3\rvert = 2$
Step 2 Magnitudes	$\lvert\mathbf{b}\rvert = \sqrt{11},\ \lvert\mathbf{n}\rvert = \sqrt{6}$
Step 3 $\sin\theta$, not $\cos\theta$	$\sin\theta = \dfrac{2}{\sqrt{66}}$: $\theta = 14.3°$

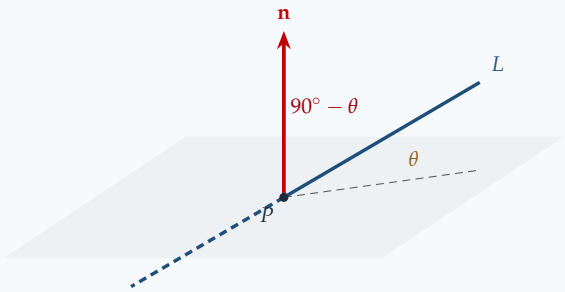


Figure 1 — The line pierces the plane at one point. Its angle with the plane is 90° minus its angle with the normal.

Watch out 1

The angle between a line and a plane uses sin, not cos: with cos you find the angle between the line and the normal.

APPLY

3 The acute angle between the planes $x - z = 0$ and $y + 2z = 1$.

I.3 The line where two planes meet

Property 3 *the line where two planes meet*

Two non-parallel planes meet in a line. Its direction is perpendicular to both normals, $\mathbf{n}_1 \times \mathbf{n}_2$; a point on it comes from setting one coordinate, say $z = 0$, and solving the two equations.

Example 3

Find a vector equation of the line where $x + y + z = 4$ and $x - y + 2z = 5$ meet.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Direction $\mathbf{n}_1 \times \mathbf{n}_2$	$\begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}3\\-1\\-2\end{pmatrix}$
Step 2 A point: set $z = 0$	$x + y = 4,\ x - y = 5$: $(4.5, -0.5, 0)$
Step 3 The equation	$\mathbf{r} = \begin{pmatrix}4.5\\-0.5\\0\end{pmatrix} + t\begin{pmatrix}3\\-1\\-2\end{pmatrix}$

APPLY

4 The direction of the line where $2x - y = 3$ and $y + z = 1$ meet.

II SYSTEMS OF THREE EQUATIONS, AND THREE PLANES

AHL 1.16 · row reduction, a unique solution, none, or infinitely many · AHL 3.18 · three planes

II.1 Solving by elimination

Property 4 *solving by elimination*

Eliminate x from two equations, then y from the two new ones (**row reduction**). The last equation decides:
$az = b$ with $a \neq 0$: **one solution**;
$0 = 0$: **infinitely many**, written with a parameter, $z = t$;
$0 = b$ with $b \neq 0$: **no solution**, the system is inconsistent.

Example 4

Solve $x + 2y + z = 8,\ 2x - y + z = 3,\ x + y - z = 0$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Eliminate x	$R_2 - 2R_1$: $-5y - z = -13$; $R_3 - R_1$: $-y - 2z = -8$
Step 2 Eliminate y	$y = 8 - 2z$: $-5(8 - 2z) - z = -13,\ 9z = 27$
Step 3 The last unknown	$z = 3$
Step 4 Back-substitute	$y = 2,\ x = 1$: one solution $(1, 2, 3)$

APPLY

1 Paper 2: $x + 2y - z = -3,\ 3x - y + z = 10,\ x + y + 2z = 7$.

II.2 Infinitely many, or none: three planes

Property 5 *three planes*

Each equation is a plane. One solution: the three planes meet at **one point**. Infinitely many with one parameter: they share **a line** (a sheaf). No solution: they have no common point (parallel planes, or a triangular prism).

Example 5

$x + y - z = 2,\ x - y + z = 0,\ 3x + y - z = k$. Find k for which there are infinitely many solutions, and the solutions.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Eliminate x	$R_2 - R_1$: $-2y + 2z = -2$; $R_3 - 3R_1$: $-2y + 2z = k - 6$
Step 2 The last row: $0 = 0$	$k - 6 = -2$: $k = 4$
Step 3 A parameter	$z = t$: $y = 1 + t,\ x = 2 - y + z = 1$
Step 4 Interpret	$(1, 1 + t, t)$: the three planes share a line

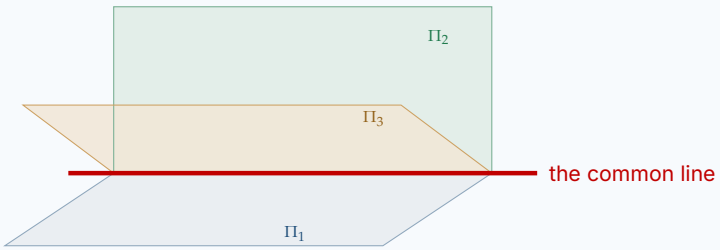


Figure 2 — Three planes through one line: the system has infinitely many solutions, one for each point of the line.

Watch out 2

$0 = 0$ does not mean "no solution": it means one equation was a combination of the others, and the solutions form a line.

APPLY

2 Row reduction ends with $0 = 3$.

3 Row reduction ends with $0 = 0$, one parameter.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A A line and a plane

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 7]

The plane Π has equation $x + 2y - 2z = 5$ and the line L has equation $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

(a) Find the coordinates of the point where L meets Π. [4]

(b) Find the sine of the acute angle between L and Π. [3]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Substitute the line	$(1 + 2\lambda) + 2(-1 + \lambda) - 2(2 + \lambda) = 5$ *Substitute the line into the plane.*	M1
Step 2 Collect	$2\lambda - 5 = 5$	A1
Step 3 Solve	$\lambda = 5$	A1
Step 4 The point	$(11, 4, 7)$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $\mathbf{b} \cdot \mathbf{n}$	$\mathbf{b} \cdot \mathbf{n} = 2 + 2 - 2 = 2$	A1
Step 2 Magnitudes	$\lvert\mathbf{b}\rvert = \sqrt{6},\ \lvert\mathbf{n}\rvert = 3$	A1
Step 3 $\sin\theta$	$\sin\theta = \dfrac{2}{3\sqrt{6}} = \dfrac{\sqrt{6}}{9}$	A1

B A system with a parameter

EXAM-STYLE QUESTION 2 Paper 1 · no calculator · [Maximum mark: 7]

Consider the system of equations

$$x + y + z = 2, \qquad 2x - y + z = 1, \qquad x + 4y + 2z = k,$$

where $k \in \mathbb{R}$.

(a) Find the value of k for which the system has infinitely many solutions. [3]

(b) For this value of k, find the general solution of the system. [3]

(c) Interpret the general solution geometrically. [1]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Eliminate x	$R_2 - 2R_1$: $-3y - z = -3$; $R_3 - R_1$: $3y + z = k - 2$ *Eliminate x.*	M1
Step 2 Add the new rows	adding: $0 = k - 5$	A1
Step 3 $0 = 0$: the value of k	$k = 5$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 A parameter, $y = t$	let $y = t$: $z = 3 - 3t$ *A parameter for one unknown.*	M1
Step 2 Back-substitute	$x = 2 - t - (3 - 3t) = -1 + 2t$	A1
Step 3 The general solution	$(x, y, z) = (-1 + 2t, t, 3 - 3t)$ *Or an equivalent form, as $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.*	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Interpret	the three planes meet in a line	A1

C Two planes on the GDC (Paper 2)

EXAM-STYLE QUESTION 3 Paper 2 · calculator · [Maximum mark: 6]

The planes Π_1 and Π_2 have equations $x - 2y + z = 3$ and $2x + y - z = 1$.

(a) Find the acute angle between Π_1 and Π_2, in degrees. [2]

(b) Find a vector in the direction of the line of intersection of Π_1 and Π_2. [2]

(c) Find a vector equation of this line. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The normals, with the modulus	$\cos\theta = \dfrac{\lvert 2 - 2 - 1\rvert}{\sqrt{6} \times \sqrt{6}} = \dfrac{1}{6}$	M1
Step 2 The angle	$80.4°$ $(80.4059\ldots)$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $\mathbf{n}_1 \times \mathbf{n}_2$	$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$	M1
Step 2 Compute	$\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 A point with $z = 0$	$z = 0$: $x - 2y = 3$, $2x + y = 1$, so $(1, -1, 0)$ *Any point of both planes.*	M1
Step 2 The equation	$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$	A1

IV WHICH METHOD? READ THE QUESTION

AHL 1.16 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"where the line meets the plane"	substitute the line	one equation in λ, then the point P1
$\mathbf{b}\cdot\mathbf{n}=0$	parallel, or in the plane	test one point of the line P1
"the angle between the line and the plane"	$\sin\theta$ with $\mathbf{b}$ and $\mathbf{n}$	$\frac{\lvert\mathbf{b}\cdot\mathbf{n}\rvert}{\lvert\mathbf{b}\rvert\lvert\mathbf{n}\rvert}$, then arcsin P2
"the angle between the planes"	the normals	$\cos\theta=\frac{\lvert\mathbf{n}_1\cdot\mathbf{n}_2\rvert}{\lvert\mathbf{n}_1\rvert\lvert\mathbf{n}_2\rvert}$ P2
"the line of intersection of the planes"	direction $\mathbf{n}_1\times\mathbf{n}_2$	a point with $z=0$, then $\mathbf{r}=\mathbf{a}+t\mathbf{d}$ P3
three equations, three unknowns	row reduction	eliminate x, then y P4
"infinitely many solutions", a parameter k	the last row becomes $0=0$	k from it, then $z=t$ or $y=t$ P4
"interpret geometrically"	one point, a line, or none	the three planes meet in... P5

In the exam 1 *command terms*

Show that a system is inconsistent: reach a false equation such as $0=3$. **General solution**: every unknown in terms of one parameter. **Interpret geometrically**: name the figure the planes share.

V ON THE GDC: PAPER 2

AHL 3.18 · 1.16 · NumWorks · every calculator move of this topic

1 A system of three equations

Equations → x-2y+z=3 → 2x+y-z=1 → z=0 → Solve

A point of the line of intersection: add the equation $z=0$ to the two planes.
On paper: the point $(1,\,-1,\,0)$.

EQUATIONS

```
x=1
y=-1
z=0
```

2 An angle between two planes

Settings → Degrees → acos(abs(dot(n,m))/(norm(n)norm(m)))

With the normals stored as `n` and `m`. **On paper:** $\cos\theta = \frac{1}{6}$, then $\theta = 80.4°$.

CALCULATIONS

```
acos(1/6)
=80.40593
```

ANSWERS TO THE APPLY QUESTIONS

I.1 $\mathbf{b}\cdot\mathbf{n} = 0$ and $2 \neq 5$: parallel, no common point

I.2 $2\lambda + 1 = 6$: $\left(\frac{5}{2},\ 5,\ \frac{7}{2}\right)$

I.3 $\cos\theta = \frac{|-2|}{\sqrt{2}\sqrt{5}}$: $50.8°$

I.4 $\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$

II.1 $(2,\ -1,\ 3)$

II.2 no solution: the planes have no common point

II.3 infinitely many: a line

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