Intersections, angles and systems

Skills sheet · HL AA · AHL 3.18 · 1.16

KEY FACTS

line and plane: substitute, one equation in λ P1

$\sin\theta = \frac{|\mathbf{b}\cdot\mathbf{n}|}{|\mathbf{b}||\mathbf{n}|}$, planes: $\cos\theta$ with normals P2

line of two planes: $\mathbf{n}_1 \times \mathbf{n}_2$, a point with $z = 0$ P3

row reduction: $az = b$, $0 = 0$, $0 = b$ P4

three planes: a point, a line, or none P5

Line and plane: sin, *not* cos W1.

$0 = 0$: *infinitely many, not none* W2.

$\mathbf{b} \cdot \mathbf{n} = 0$: *parallel or in the plane* P1.

WHAT UNLOCKS

- "where the line meets the plane" $\rightarrow$ the line into the plane.
- "the line of intersection" $\rightarrow$ $\mathbf{n}_1 \times \mathbf{n}_2$ and one common point.
- "infinitely many solutions" $\rightarrow$ make the last row $0 = 0$.
- "interpret geometrically" $\rightarrow$ the figure the planes share.

WORKED EXAMPLE ①

Does $\mathbf{r} = \begin{pmatrix}0\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$ **meet** $x + y + 2z = 3$**?** no GDC

$\mathbf{b} \cdot \mathbf{n} = 1 + 1 - 2 = 0$ M1

$(0, 1, 2)$: $5 \neq 3$, so parallel, no point A1

Method

Line and plane

① $\mathbf{b} \cdot \mathbf{n}$ first.
② If 0, test one point.
③ Else substitute.

WORKED EXAMPLE ②

Solve $x + y = 3$, $y + z = 5$, $x + z = 4$. no GDC

adding: $x + y + z = 6$ M1

$(1, 2, 3)$ A1

Method

A small system

① Combine the equations.
② Find one unknown.
③ Back-substitute and check.

FLUENCY

●○○ **1.** Where does $\mathbf{r} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\3\end{pmatrix}$ meet $z = 6$? no GDC

●○○ **2.** Angle between the planes $x + y = 1$ and $y + z = 2$ no GDC

●○○ **3.** Direction of the line of $x + y + z = 1$ and $x - y = 0$ no GDC

●○○ **4.** Angle between direction $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and the plane $x = 5$ no GDC

●○○ **5.** Solve $x + y + z = 6$, $y + z = 5$, $z = 3$ no GDC

●○○ **6.** The last row reads $0 = 4$. Conclusion? no GDC

REASONING

●●● **Question 1.** The line L: $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and the plane Π: $3x - y + z = 4$. [no GDC]

(a) Show that L is parallel to Π and does not lie in Π. [3]

●●● **Question 2.** The planes $x + y + z = 6$ and $x - y + 2z = 3$ are given. [no GDC]

(a) Find a vector equation of their line of intersection. [4]

●●● **Question 3.** The line $\mathbf{r} = t \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and the plane $2x - y + 2z = 0$ are given. [with GDC]

(a) Find the acute angle between them. [3]

PROBLEMS

●●● **Question 4.** Consider the system $x + 2y + z = 4,\ 2x + y - z = 1,\ x - y + az = b$. [no GDC]

(a) Find the conditions on a and b for the system to have no solution, and for it to have infinitely many. [5]

●●● **Question 5.** Three planes have equations $2x - y + 3z = 7.4,\ x + 4y - z = -3.1$ and $3x + 2y + 2z = 6.5$. [with GDC]

(a) Find their point of intersection. [2]

ANSWERS

1 $\lambda = 2$: $(3, 4, 6)$

2 $\cos\theta = \frac{1}{2}$, $60°$

3 $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

4 $\sin\theta = \frac{1}{\sqrt{2}}$, $45°$

5 $(1, 2, 3)$

6 no solution: the system is inconsistent

Q1 $6 + 3\lambda + 1 - 2\lambda + 1 - \lambda = 8 \neq 4$ for every λ: no common point; $\mathbf{b} \cdot \mathbf{n} = 3 - 2 - 1 = 0$: parallel, not in Π

Q2 $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$; $z = 0$: $(4.5, 1.5, 0)$: $\mathbf{r} = \begin{pmatrix} 4.5 \\ 1.5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$

Q3 $\sin\theta = \frac{|2-2+4|}{3\times 3} = \frac{4}{9}$, $\theta = 26.4°$

Q4 $R_2 - 2R_1$: $-3y - 3z = -7$; $R_3 - R_1$: $-3y + (a-1)z = b - 4$; subtract: $(a+2)z = b + 3$. No solution: $a = -2$, $b \neq -3$; infinitely many: $a = -2$, $b = -3$

Q5 $(4.46, -2.2, -1.24)$

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