The vector product

The scalar product turns two vectors into a number. The vector product turns them into a third vector, perpendicular to both, whose length is the area of the parallelogram they span. That one vector answers two kinds of question: which direction is perpendicular to both (a normal, a common perpendicular), and how large is this triangle or parallelogram.

I A VECTOR PERPENDICULAR TO TWO VECTORS

AHL 3.16 · $\mathbf{v} \times \mathbf{w}$, perpendicular to both, $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$

I.1 The vector product

Definition 1 *the vector product*

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}.$$

It is a **vector**, perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, in the direction given by the right-hand rule. The formula is in the formula booklet.

Example 1

Find $\mathbf{v} \times \mathbf{w}$ for $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$, and check it.

WHAT YOU DO	WHAT YOU WRITE
Step 1 First component: $v_2w_3 - v_3w_2$	$1 \times 4 - (-3)(-2) = -2$
Step 2 Second: $v_3w_1 - v_1w_3$	$(-3)(1) - 2 \times 4 = -11$
Step 3 Third: $v_1w_2 - v_2w_1$	$2(-2) - 1 \times 1 = -5$: $\mathbf{v} \times \mathbf{w} = \begin{pmatrix} -2 \\ -11 \\ -5 \end{pmatrix}$
Step 4 Check: two scalar products	$-4 - 11 + 15 = 0$, $-2 + 22 - 20 = 0$

Watch out 1

The middle component is $v_3w_1 - v_1w_3$, not $v_1w_3 - v_3w_1$. Check the answer: its scalar product with $\mathbf{v}$ and with $\mathbf{w}$ must both be 0.

APPLY

1 $\begin{pmatrix}1\\0\\1\end{pmatrix} \times \begin{pmatrix}0\\2\\0\end{pmatrix}$.

2 A unit vector perpendicular to $\begin{pmatrix}2\\0\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\2\end{pmatrix}$.

I.2 Order, and parallel vectors

Property 1 *rules*

$\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$: the order matters. $\mathbf{v} \times \mathbf{v} = \mathbf{0}$, and two non-zero vectors are **parallel** if and only if $\mathbf{v} \times \mathbf{w} = \mathbf{0}$. The vector product distributes: $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.

Example 2

$\mathbf{a} \times \mathbf{b} = \begin{pmatrix}3\\-1\\2\end{pmatrix}$. Find $\mathbf{b} \times \mathbf{a}$ and $(2\mathbf{a}) \times \mathbf{b}$. Are $\begin{pmatrix}2\\-4\\6\end{pmatrix}$ and $\begin{pmatrix}-1\\2\\-3\end{pmatrix}$ parallel?

WHAT YOU DO	WHAT YOU WRITE
Step 1 The order changes the sign	$\mathbf{b} \times \mathbf{a} = \begin{pmatrix}-3\\1\\-2\end{pmatrix}$
Step 2 A number comes out	$(2\mathbf{a}) \times \mathbf{b} = 2\,\mathbf{a} \times \mathbf{b} = \begin{pmatrix}6\\-2\\4\end{pmatrix}$
Step 3 Parallel: the product is $\mathbf{0}$	$\begin{pmatrix}2\\-4\\6\end{pmatrix} = -2\begin{pmatrix}-1\\2\\-3\end{pmatrix}$, so the product is $\mathbf{0}$: parallel

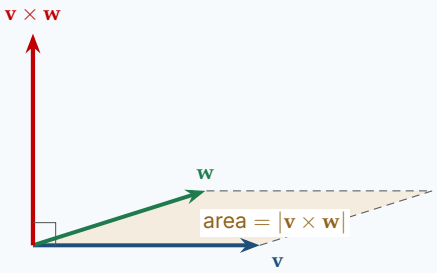


Figure 1 — $\mathbf{v}$ and $\mathbf{w}$ lie flat; $\mathbf{v} \times \mathbf{w}$ stands up from their plane, by the right-hand rule. Swap the order and it points down.

APPLY

3 Are $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}2\\4\\5\end{pmatrix}$ parallel?

II.1 Areas

Property 2 *the magnitude is an area*

$$|\mathbf{v}\times\mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}|\sin\theta,$$

the area of the parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$. So the area of the triangle ABC is

$$\tfrac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|.$$

Example 3

Find the area of the triangle with vertices $A(2,\,1,\,0)$, $B(4,\,1,\,1)$ and $C(3,\,3,\,0)$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Two vectors from one vertex	$\overrightarrow{AB} = \begin{pmatrix}2\\0\\1\end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix}1\\2\\0\end{pmatrix}$
Step 2 The vector product	$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{pmatrix}-2\\1\\4\end{pmatrix}$
Step 3 Half the magnitude	$\tfrac{1}{2}\sqrt{4+1+16} = \dfrac{\sqrt{21}}{2}$

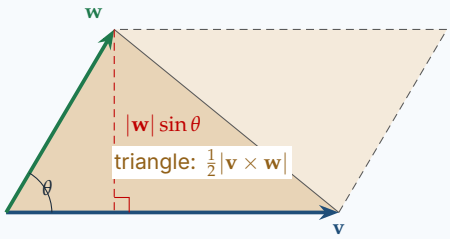


Figure 2 — The parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$ has area $|\mathbf{v}\times\mathbf{w}|$; the triangle is half of it.

APPLY

1 The parallelogram with sides $\begin{pmatrix}1\\2\\2\end{pmatrix}$ and $\begin{pmatrix}2\\0\\-1\end{pmatrix}$: its area.

II.2 The distance from a point to a line

Property 3 *the distance from a point to a line*

The height of the parallelogram above the side $\overrightarrow{AB}$ is its area divided by the base: the distance from C to the line AB is

$$\frac{\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|}{\left|\overrightarrow{AB}\right|}.$$

Example 4

Find the distance from $C(1, 4, 2)$ to the line through $A(2, 0, 1)$ and $B(4, 1, 3)$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Vectors from A	$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$
Step 2 The vector product	$\begin{pmatrix} -7 \\ -4 \\ 9 \end{pmatrix}$, magnitude $\sqrt{146}$
Step 3 Area over base $\left\|\overrightarrow{AB}\right\| = 3$.	$\dfrac{\sqrt{146}}{3} = 4.03$

Watch out 2

Both vectors must leave the **same** vertex: $\overrightarrow{AB}$ and $\overrightarrow{AC}$, never $\overrightarrow{AB}$ and $\overrightarrow{BC}$ followed blindly (that gives the same area here, but not the same angle).

APPLY

2 The distance from $(0, 0, 5)$ to the line through O with direction $\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A A common perpendicular

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 6]

Let $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$.

(a) Find $\mathbf{a} \times \mathbf{b}$. [2]

(b) Find a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. [2]

(c) Find a vector equation of the line through $(0, 1, 2)$ perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The three components	$\begin{pmatrix} 2\times 2-(-1)\times 0 \\ (-1)\times 3-1\times 2 \\ 1\times 0-2\times 3 \end{pmatrix}$ *Correct method for at least two components.*	M1
Step 2 Simplify	$\begin{pmatrix} 4 \\ -5 \\ -6 \end{pmatrix}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Magnitude	$\lvert\mathbf{a}\times\mathbf{b}\rvert = \sqrt{16+25+36} = \sqrt{77}$	M1
Step 2 Divide by it	$\frac{1}{\sqrt{77}}\begin{pmatrix} 4 \\ -5 \\ -6 \end{pmatrix}$ *Or its opposite.*	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Direction $\mathbf{a}\times\mathbf{b}$	direction $\mathbf{a}\times\mathbf{b}$	M1
Step 2 The equation	$\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 4 \\ -5 \\ -6 \end{pmatrix}$	A1

B The area of a triangle

EXAM-STYLE QUESTION 2 Paper 1 · no calculator · [Maximum mark: 6]

The points $A(1, 0, 1)$, $B(3, 1, 1)$ and $C(1, 2, 3)$ are the vertices of a triangle.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [2]

(b) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. [2]

(c) Find the area of the triangle ABC. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $\overrightarrow{AB}$	$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$	A1
Step 2 $\overrightarrow{AC}$	$\overrightarrow{AC} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The three components	$\begin{pmatrix} 1 \times 2 - 0 \times 2 \\ 0 \times 0 - 2 \times 2 \\ 2 \times 2 - 1 \times 0 \end{pmatrix}$	M1
Step 2 Simplify	$\begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Half the magnitude	$\frac{1}{2}\sqrt{4 + 16 + 16}$	M1
Step 2 The area	$\text{area} = 3$	A1

C On the GDC (Paper 2)

EXAM-STYLE QUESTION 3 Paper 2 · calculator · [Maximum mark: 6]

The points $P(1.5,\ -2,\ 0)$, $Q(4,\ 1,\ 2.5)$ and $R(-1,\ 3,\ 1)$ are the vertices of a triangle.

(a) Find $\overrightarrow{PQ} \times \overrightarrow{PR}$. [2]

(b) Find the area of the triangle PQR. [2]

(c) Find the shortest distance from R to the line PQ. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Vectors from P, then the product	$\overrightarrow{PQ} = \begin{pmatrix} 2.5 \\ 3 \\ 2.5 \end{pmatrix}$, $\overrightarrow{PR} = \begin{pmatrix} -2.5 \\ 5 \\ 1 \end{pmatrix}$	M1
Step 2 Compute	$\begin{pmatrix} -9.5 \\ -8.75 \\ 20 \end{pmatrix}$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Half the magnitude	$\frac{1}{2}\left\lvert \overrightarrow{PQ} \times \overrightarrow{PR} \right\rvert = \frac{1}{2} \times 23.807\ldots$	M1
Step 2 Compute	$11.9\ \ (11.9039\ldots)$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Area over base	$\dfrac{23.807\ldots}{\sqrt{21.5}}$ *Area of the parallelogram over the base* $\left\lvert \overrightarrow{PQ} \right\rvert$.	M1
Step 2 Compute	$5.13\ \ (5.13452\ldots)$	A1

IV WHICH METHOD? READ THE QUESTION

AHL 3.16 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"a vector perpendicular to both $\mathbf{a}$ *and* $\mathbf{b}$*"*	$\mathbf{a} \times \mathbf{b}$	the three components, then check with two scalar products D1
"a unit vector perpendicular to both"	divide by $\lvert\mathbf{a} \times \mathbf{b}\rvert$	$\frac{1}{\lvert\mathbf{a}\times\mathbf{b}\rvert}\,\mathbf{a} \times \mathbf{b}$ D1
"are the vectors parallel?"	$\mathbf{v} \times \mathbf{w} = \mathbf{0}$	or $\mathbf{w} = k\mathbf{v}$ P1
$\mathbf{b} \times \mathbf{a}$ *from* $\mathbf{a} \times \mathbf{b}$	the order changes the sign	$-\,\mathbf{a} \times \mathbf{b}$ P1
"the area of the triangle ABC*"*	$\frac{1}{2}\left\lvert\overrightarrow{AB} \times \overrightarrow{AC}\right\rvert$	two vectors from one vertex P2
"the area of the parallelogram"	$\lvert\mathbf{v} \times \mathbf{w}\rvert$	no half P2
"the distance from C *to the line* AB*"*	area divided by base	$\frac{\lvert\overrightarrow{AB}\times\overrightarrow{AC}\rvert}{\lvert\overrightarrow{AB}\rvert}$ P3
a line perpendicular to two given directions	direction $\mathbf{b}_1 \times \mathbf{b}_2$	$\mathbf{r} = \mathbf{a} + t\,\mathbf{b}_1 \times \mathbf{b}_2$ D1

In the exam 1 *command terms*

Find $\mathbf{a} \times \mathbf{b}$: the working of at least two components earns the method mark. **Hence**: use the vector just found. Areas are exact in Paper 1 ($\sqrt{77}$, not 8.77).

V ON THE GDC: PAPER 2

AHL 3.16 · NumWorks · every calculator move of this topic

1 The vector product

Calculation → cross([[2.5][3][2.5]],[[-2.5][5][1]])

`cross` gives the vector product of two column vectors, in that order. **On paper:** the two vectors, then the product.

CALCULATIONS

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[[-9.5][-8.75][20]]
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2 An area

Calculation → norm(Ans)/2

`Ans` reuses the last answer: half its norm is the area of the triangle. **On paper:** $\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$, then the area to three significant figures.

CALCULATIONS

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norm(Ans)/2
=11.90391
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ANSWERS TO THE APPLY QUESTIONS

I.1 $\begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$

I.2 $\frac{1}{\sqrt{21}}\begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix}$

I.3 no: the product is $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \neq \mathbf{0}$

II.1 $\left|\begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix}\right| = \sqrt{45} = 3\sqrt{5}$

II.2 $\dfrac{|(15,\,0,\,0)|}{5} = 3$

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