Vector equations of lines

In space a line has no gradient and no equation $y = mx + c$. It has a point to start from and a direction to follow: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. Every point of the line is one value of λ, which makes the questions concrete: is this point on the line, where does it cross that plane, do these two lines meet, and where is a moving object at time t?

I THE EQUATION OF A LINE

AHL 3.14 · $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, parametric and Cartesian forms, $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$

I.1 The vector equation of a line

Definition 1 *the vector equation of a line*

The line through the point with position vector $\mathbf{a}$, with **direction vector $\mathbf{b}$**, is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}, \qquad \lambda \in \mathbb{R}.$$

Each value of λ gives one point. Through A and B: $\mathbf{r} = \mathbf{a} + \lambda\overrightarrow{AB}$.

Example 1

Find a vector equation of the line through $A(3, 0, -1)$ and $B(5, -1, 1)$, and the point with $\lambda = -1$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Direction $\overrightarrow{AB}$	$\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$
Step 2 A point, then the direction	$\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$
Step 3 Substitute $\lambda = -1$	$(1, 1, -3)$

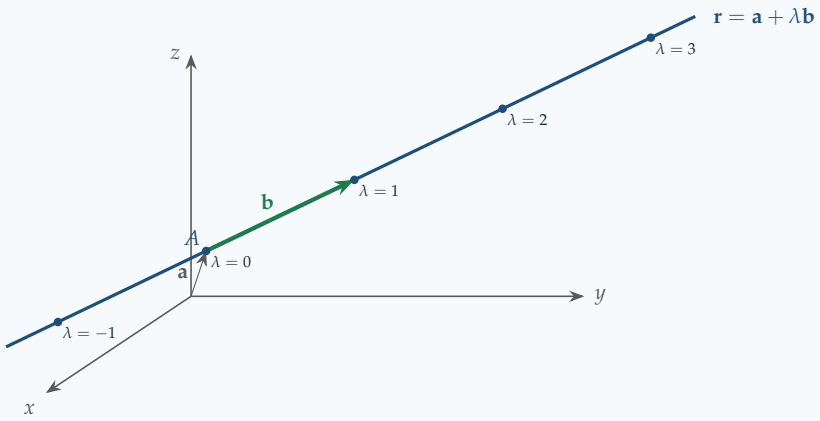


Figure 1 — The line through A with direction $\mathbf{b}$: $\lambda = 0$ at A, $\lambda = 1$ one step of $\mathbf{b}$ further, $\lambda < 0$ behind A.

APPLY

1 The line through $(0,\ 4,\ 1)$ parallel to $\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$.

2 $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$: the point with $\lambda = 2$.

I.2 Parametric and Cartesian forms; testing a point

Property 1 *parametric and Cartesian forms*

Component by component, $x = a_1 + \lambda b_1$, $y = a_2 + \lambda b_2$, $z = a_3 + \lambda b_3$; solving each for λ gives the Cartesian form

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}.$$

A point is on the line when **one** value of λ gives all three coordinates.

Example 2

$L : \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$. Write L in Cartesian form, and decide whether $P(9,\ -3,\ 7)$ lies on L.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Cartesian form	$\frac{x-3}{2} = \frac{y}{-1} = \frac{z+1}{2}$
Step 2 λ from one coordinate	$3 + 2\lambda = 9$: $\lambda = 3$
Step 3 Check the other two	$y = -3$ yes; $z = -1 + 6 = 5 \neq 7$
Step 4 Conclude	P is not on L

Watch out 1

To test a point, find λ from one coordinate and **check it in the other two**. Two coordinates that agree are not enough.

APPLY

3 $\frac{x-1}{4} = \frac{y+2}{3} = \frac{z}{-1}$: a point and a direction.

4 Where $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ meets $z = 0$.

I.3 A moving object

Property 2 *a moving object*

An object at $\mathbf{r}_0$ at time $t = 0$, moving with constant velocity $\mathbf{v}$, is at

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

at time t. Its **speed** is $|\mathbf{v}|$. It is closest to a point T when $\overrightarrow{TP} \cdot \mathbf{v} = 0$.

Example 3

A boat is at $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ (km) after t hours. Find its speed, and the shortest distance to the buoy $T(10, 6, 0)$.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Speed $= \lvert\mathbf{v}\rvert$	$\sqrt{9+16} = 5$ km h^{-1}
Step 2 $\overrightarrow{TP}$ at time t	$\begin{pmatrix} 3t-10 \\ 4t-5 \\ 0 \end{pmatrix}$
Step 3 Closest: $\overrightarrow{TP} \cdot \mathbf{v} = 0$	$9t - 30 + 16t - 20 = 0$: $t = 2$
Step 4 The distance	$\overrightarrow{TP} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}$: 5 km

APPLY

5 The speed for $\mathbf{r} = \mathbf{r}_0 + t \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

6 $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 0.5 \end{pmatrix}$: the position at $t = 3$.

II TWO LINES: PARALLEL, MEETING OR SKEW

AHL 3.15 · coincident, parallel, intersecting, skew; the angle between two lines

II.1 Parallel, intersecting or skew

Property 3 *four cases*

Two lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$ are
parallel when $\mathbf{b}_2 = k\mathbf{b}_1$ (and **coincident** if they share a point);
intersecting when one pair (λ, μ) gives the same point;
skew when they are neither parallel nor meeting, which only happens in space.

Example 4

$L_1 : \mathbf{r} = \begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix}3\\0\\1\end{pmatrix} + \mu\begin{pmatrix}2\\1\\1\end{pmatrix}$. Show that the lines are skew.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Parallel?	$\begin{pmatrix}2\\1\\1\end{pmatrix}$ is not a multiple of $\begin{pmatrix}1\\-1\\2\end{pmatrix}$: not parallel
Step 2 Equate the coordinates	$1+\lambda = 3+2\mu,\ 2-\lambda = \mu,\ 2\lambda = 1+\mu$
Step 3 Solve two of them	second and third: $\lambda = 1, \mu = 1$
Step 4 Test the third	first: $2 \neq 5$: they do not meet, so skew

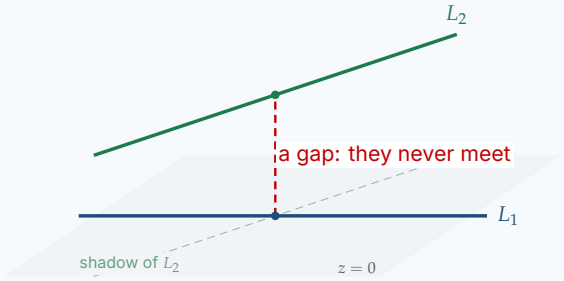


Figure 2 — Two lines in space: seen from above they cross, but at the crossing they are at different heights. They are skew.

APPLY

1 $\mathbf{r} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\4\\-2\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}0\\1\\1\end{pmatrix} + \mu\begin{pmatrix}-1\\-2\\1\end{pmatrix}$.

II.2 The point of intersection, and the angle

Property 4 *finding the point, and the angle*

Equate the two position vectors: three equations in λ and μ. Solve two of them, then **test the third**: if it holds, the lines meet; if not, they are skew. The acute angle between the lines is the angle between their directions:

$$\cos\theta = \frac{|\mathbf{b}_1 \cdot \mathbf{b}_2|}{|\mathbf{b}_1|\,|\mathbf{b}_2|}.$$

Example 5

$L_1 : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$. Find their point of intersection and the acute angle between them.

WHAT YOU DO	WHAT YOU WRITE
Step 1 Equate the coordinates	$2+\lambda = 1+\mu,\ 1+\lambda = -2+2\mu,\ -1 = 3-2\mu$
Step 2 Solve two of them	third: $\mu = 2$; first: $\lambda = 1$
Step 3 Test the third	second: $1+1=2$ and $-2+4=2$: they meet at $(3,\,2,\,-1)$
Step 4 The angle from the directions	$\cos\theta = \dfrac{\lvert 1+2+0\rvert}{\sqrt{2}\times 3} = \dfrac{1}{\sqrt{2}}$: $\theta = 45°$

Watch out 2

Use two different letters, λ and μ: the two lines do not reach their common point at the same value of the parameter.

APPLY

2 The acute angle between directions $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

A One line

EXAM-STYLE QUESTION 1 Paper 1 · no calculator · [Maximum mark: 6]

The line L passes through the points $A(1, -2, 4)$ and $B(3, 1, 2)$.

(a) Find a vector equation of L. [2]

(b) Show that the point $C(7, 7, -2)$ lies on L. [2]

(c) Find the coordinates of the point where L meets the plane $z = 0$. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Direction $\overrightarrow{AB}$	$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$	A1
Step 2 The equation	$\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$	A1

Any point of the line and any multiple of the direction.

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 λ from one coordinate	$1 + 2\lambda = 7 \Rightarrow \lambda = 3$	M1
Step 2 Check the other two	$y = -2 + 9 = 7,\ z = 4 - 6 = -2$, so C is on L	R1

Both other coordinates checked. AG.

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $z = 0$ gives λ	$4 - 2\lambda = 0 \Rightarrow \lambda = 2$	M1
Step 2 The point	$(5, 4, 0)$	A1

B Two lines that meet

EXAM-STYLE QUESTION 2 Paper 1 · no calculator · [Maximum mark: 7]

The lines L_1 and L_2 have equations $\mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}2\\3\\-1\end{pmatrix} + \mu\begin{pmatrix}0\\2\\-2\end{pmatrix}$.

(a) Show that L_1 and L_2 intersect, and find the coordinates of their point of intersection. [5]

(b) Find the cosine of the acute angle between L_1 and L_2. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Equate the coordinates	$1+\lambda = 2,\ -\lambda = 3+2\mu,\ 2+\lambda = -1-2\mu$ *Equate the components.*	M1
Step 2 Solve for λ	$\lambda = 1$	A1
Step 3 Solve for μ	$\mu = -2$	A1
Step 4 Test the third equation	z: $2+1 = 3$ and $-1+4 = 3$: the third equation holds, so the lines meet	R1
Step 5 The point	$(2, -1, 3)$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 The directions, with the modulus	$\dfrac{\lvert 0-2-2\rvert}{\sqrt{3}\times\sqrt{8}}$ *The directions only.*	M1
Step 2 Simplify	$\cos\theta = \dfrac{4}{\sqrt{24}} = \dfrac{\sqrt{6}}{3}$	A1

C A moving object (Paper 2)

EXAM-STYLE QUESTION 3 Paper 2 · calculator · [Maximum mark: 6]

A drone flies in a straight line. At time t hours its position vector, in kilometres, is $\mathbf{r} = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 0.5 \end{pmatrix}$. A tower has its top at $T(5, 8, 1)$.

(a) Find the speed of the drone. [2]

(b) Find the value of t when the drone is closest to T. [2]

(c) Find the shortest distance between the drone and T. [2]

WORKING (a)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Speed $= \lvert\mathbf{v}\rvert$	$\sqrt{2^2 + 1^2 + 0.5^2}$	M1
Step 2 Compute	2.29 km h^{-1} $(2.29128\ldots)$	A1

WORKING (b)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $\overrightarrow{TP} \cdot \mathbf{v} = 0$	$\overrightarrow{TP} = \begin{pmatrix} 2t-9 \\ t-6 \\ 0.5t \end{pmatrix}$, $\overrightarrow{TP} \cdot \begin{pmatrix} 2 \\ 1 \\ 0.5 \end{pmatrix} = 5.25t - 24 = 0$	M1
	Or minimise the distance on the GDC.	
Step 2 Solve	$t = 4.57$ $(\frac{32}{7})$	A1

WORKING (c)

WHAT YOU DO	WHAT YOU WRITE	
Step 1 $\lvert\overrightarrow{TP}\rvert$ at that t	$\lvert\overrightarrow{TP}\rvert$ at $t = \frac{32}{7}$	M1
Step 2 Compute	2.70 km $(2.69920\ldots)$	A1

IV WHICH METHOD? READ THE QUESTION

3.15 · the whole topic on one page

THE QUESTION SAYS	YOU KNOW	YOU WRITE
"a vector equation of the line through A and B"	$\mathbf{r} = \mathbf{a} + \lambda\overrightarrow{AB}$	a point, then the direction D1
"show that C lies on the line"	one λ for all three	λ from one coordinate, check the other two W1
"where the line meets the plane $z = 0$"	the parametric form	$z = 0$ gives λ, then the point P1
a Cartesian equation $\frac{x-a_1}{b_1} = \ldots$	the point and the direction	read $\mathbf{a}$ and $\mathbf{b}$ from it P1
a position $\mathbf{r}_0 + t\mathbf{v}$	speed $= \lvert\mathbf{v}\rvert$	closest when $\overrightarrow{TP} \cdot \mathbf{v} = 0$ P2
"do the lines intersect?"	λ and μ	solve two equations, test the third P4
"are the lines skew?"	not parallel and not meeting	directions not multiples, and the third equation fails P3
"the acute angle between the lines"	$\lvert\mathbf{b}_1 \cdot \mathbf{b}_2\rvert$	the directions only, with the modulus P4

In the exam 1 *command terms*

Show that a point is on a line: one value of λ, every coordinate checked. **Show that** lines intersect: the third equation checked and stated. **Skew**: not parallel and no common point, both said.

V ON THE GDC: PAPER 2

AHL 3.14 · 3.15 · NumWorks · every calculator move of this topic

1 Solve for λ and μ

Equations → x+0y=1 → -x-2y=3 → Solve

With x for λ and y for μ, solve two of the component equations; then test the third by hand. **On paper:** the three equations, the two values, and the check.

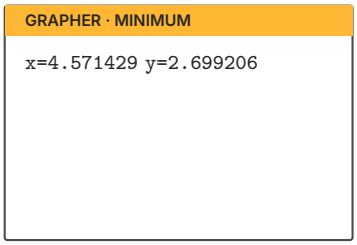
EQUATIONS

x=1
y=−2

2 The closest approach

Grapher → f(x)=√((2x-9)²+(x-6)²+(0.5x)²) → Calculate → Minimum

Draw the distance as a function of t and read its minimum. **On paper:** the distance as a function of t, then t and the distance to three significant figures.



ANSWERS TO THE APPLY QUESTIONS

I.1 $\mathbf{r} = \begin{pmatrix}0\\4\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\-3\end{pmatrix}$

I.2 $(5, -5, 1)$

I.3 $(1, -2, 0)$; $\begin{pmatrix}4\\3\\-1\end{pmatrix}$

I.4 $\lambda = 2$: $(4, 3, 0)$

I.5 3

I.6 $(-2, 6, 3.5)$

II.1 parallel, not coincident: $\begin{pmatrix}-1\\1\\1\end{pmatrix}$ is not a multiple of the direction

II.2 $\cos\theta = \frac{|-1|}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$: $60°$

Written and verified by a practising IB mathematics teacher, produced with AI assistance; every result recomputed by software, every figure computed from its function. Not affiliated with or endorsed by the International Baccalaureate Organization. © 2026 Hence