Vector equations of lines

Skills sheet · HL AA · AHL 3.14 · 3.15

KEY FACTS

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ D1

$\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$ P1

$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, speed $|\mathbf{v}|$ P2

parallel, intersecting or skew P3

$\cos\theta = \frac{|\mathbf{b}_1 \cdot \mathbf{b}_2|}{|\mathbf{b}_1||\mathbf{b}_2|}$ P4

A point: one λ for all three coordinates W1.
Two lines: λ and μ, never one letter W2.
Skew: not parallel, no common point P3.

WHAT UNLOCKS

- two points $\rightarrow$ the first as $\mathbf{a}$, their difference as $\mathbf{b}$.
- "meets the plane $z = 0$" $\rightarrow$ $z = 0$ gives λ.
- "do the lines meet?" $\rightarrow$ solve two equations, test the third.
- "closest to T" $\rightarrow$ $\overrightarrow{TP} \cdot \mathbf{v} = 0$.

WORKED EXAMPLE ①

Write $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ in Cartesian form. no GDC

$x = 2 + 4\lambda, y = -1 + \lambda, z = 3 - 2\lambda$ M1

$\frac{x-2}{4} = \frac{y+1}{1} = \frac{z-3}{-2}$ A1

Method

Cartesian form

① The parametric equations.
② Each one solved for λ.
③ Set equal.

WORKED EXAMPLE ②

Do $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ meet? no GDC

y: $1 = 2 - \mu$, $\mu = 1$; x: $1 + \lambda = 1$, $\lambda = 0$ M1

z: $0 \neq 1$; not parallel: skew A1

Method

Meet or skew?

① The easiest two equations.
② λ and μ.
③ Test the third; then the directions.

FLUENCY

●○○ **1.** Line through $(1, 0, -2)$, parallel to $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ no GDC

●○○ **2.** Line through $A(2, 2, 1)$ and $B(5, 0, 3)$ no GDC

●○○ **3.** Is $(7, -2, 6)$ on $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$? no GDC

●○○ **4.** Point and direction of $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z}{5}$ no GDC

●○○ **5.** Speed for $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ no GDC

●○○ **6.** $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$: parallel? no GDC

REASONING

●●● **Question 1.** The line L passes through $(3,\ -1,\ 2)$ and $(1,\ 1,\ 4)$. [no GDC]

(a) Find the coordinates of the point where L meets the plane $y = 0$. [3]

●●● **Question 2.** Two lines have directions $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$. [no GDC]

(a) Find the cosine of the acute angle between them. [3]

●●● **Question 3.** L_1: $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and L_2: $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. [no GDC]

(a) Show that L_1 and L_2 are skew. [4]

PROBLEMS

●●● **Question 4.** Two ships move in a plane. At time t hours their positions, in km, are $\mathbf{r}_A = t \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{r}_B = \begin{pmatrix} 10 \\ -5 \end{pmatrix} + t \begin{pmatrix} -1 \\ 6 \end{pmatrix}$. [no GDC]

(a) Show that the ships collide, and find when and where. [5]

●●● **Question 5.** A particle moves along $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. [with GDC]

(a) Find its shortest distance from the point $T(4,\ 2,\ 6)$. [4]

ANSWERS

1 $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$

2 $\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$

3 yes, $\lambda = 3$

4 $(1,\ -3,\ 0)$ and $\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$

5 $\sqrt{4+4+1} = 3$

6 parallel ($-\frac{1}{2}$ times); not coincident: $(0,\ 0,\ 1)$ is not on the first

Q1 $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$; $-1 + 2\lambda = 0$, $\lambda = \frac{1}{2}$: $(2,\ 0,\ 3)$

Q2 $\frac{|2-2+4|}{3\times 3} = \frac{4}{9}$

Q3 x: $1 + \lambda = 0$, $\lambda = -1$; y: $-1 = 1 + \mu$, $\mu = -2$; z: $0 \neq 1 + \mu = -1$: no common point; $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \neq k \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$: not parallel, so skew

Q4 $3s = 10 - u$, $4s = -5 + 6u$ give $s = u = 2.5$: the same time, so they collide, at $t = 2.5$ h, at $(7.5,\ 10)$

Q5 $\overrightarrow{TP} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 6t - 16 = 0$, $t = \frac{8}{3}$; distance $\frac{\sqrt{165}}{3} = 4.28$

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