# Vectors and the scalar product



A vector is a move: so far along $x$, so far along $y$, so far along $z$. Two points give the vector between them, a vector gives a length, and the scalar product of two vectors gives the angle between them in one line. Everything in the rest of the topic, lines, planes and their angles, is built on these few operations.

## I VECTORS IN TWO AND THREE DIMENSIONS

AHL 3.12 · components, $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, $|\mathbf{v}|$, unit and parallel vectors

### I.1 Components and the vector $\overrightarrow{AB}$

**Definition 1** *a vector and its components*

A **vector** has a size and a direction. In space it is written with its components,

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the base vectors along the three axes. The **position vector** of the point $A$ is $\mathbf{a} = \overrightarrow{OA}$, and

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}.$$

**Example 1**

$A(1, -2, 4)$ and $B(5, 0, 2)$. Find $\overrightarrow{AB}$, as a column and with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** End minus start | $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ |
| **Step 2** Subtract the components | $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ |

**APPLY**

1 $P(0, 3, -1)$, $Q(2, -1, 5)$: $\overrightarrow{PQ}$.

2 $3\mathbf{i} - \mathbf{k}$ as a column.

## I.2 Magnitude and unit vectors

**Definition 2** *magnitude and unit vector*

The **magnitude** of $\mathbf{v}$ is its length,

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2},$$

so the distance $AB$ is $\left|\overrightarrow{AB}\right|$. A **unit vector** has magnitude 1: the unit vector in the direction of $\mathbf{v}$ is $\dfrac{1}{|\mathbf{v}|}\mathbf{v}$.

**Example 2**

Find $\left|\overrightarrow{AB}\right|$ and the unit vector in the direction of $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Magnitude | $\sqrt{16+4+4} = \sqrt{24} = 2\sqrt{6}$ |
| **Step 2** Divide by the magnitude | $\dfrac{1}{2\sqrt{6}}\begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix}$ |

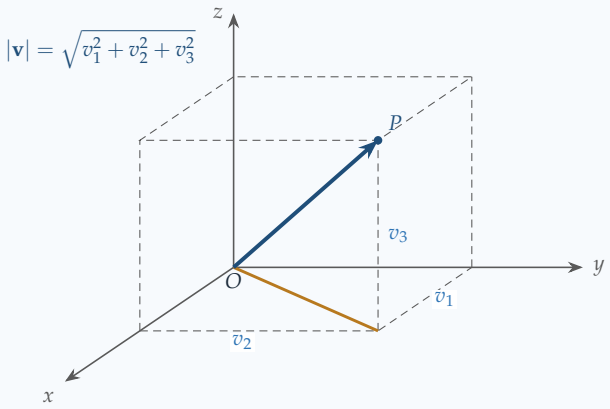



*Figure 1 — The vector $\overrightarrow{OP}$ is the diagonal of a box with sides $v_1$, $v_2$, $v_3$; Pythagoras twice gives its length.*

**APPLY**

**3** $\left|\begin{pmatrix} 1 \\ -4 \\ 8 \end{pmatrix}\right|$ and the unit vector.

**4** The distance from $(0, 1, 3)$ to $(-1, 3, 5)$.

## I.3 Parallel vectors, collinear points, parallelograms

**Property 1** *operations and parallel vectors*

Vectors add and are multiplied by a number component by component. $\mathbf{v}$ and $\mathbf{w}$ are **parallel** when $\mathbf{w} = k\mathbf{v}$ for a number $k \neq 0$; then $A$, $B$, $C$ are **collinear** when $\overrightarrow{AC} = k\overrightarrow{AB}$. In a parallelogram $ABCD$, $\overrightarrow{AD} = \overrightarrow{BC}$, so $\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}$.

**Example 3**

Show that $A(2, 0, 1)$, $B(4, 1, -2)$ and $C(6, 2, -5)$ are collinear.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** $\overrightarrow{AB}$ | $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ |
| **Step 2** $\overrightarrow{AC}$ | $\begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix} = 2\overrightarrow{AB}$ |
| **Step 3** Parallel, and a common point | $\overrightarrow{AC} \parallel \overrightarrow{AB}$ and both pass through $A$: collinear |

**Watch out 1**

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, the end minus the start, never $\mathbf{a} - \mathbf{b}$. And a parallelogram $ABCD$ is read in order round the shape: $D$ is opposite $B$.

**APPLY**

**5** Parallelogram $PQRS$, $P(1, 2, 0)$, $Q(4, 0, 1)$, $R(6, 3, 5)$: $S$.

**6** $\begin{pmatrix} 2 \\ k \\ -6 \end{pmatrix}$ parallel to $\begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$: $k$.

# II THE SCALAR PRODUCT AND THE ANGLE BETWEEN TWO VECTORS

AHL 3.13 · $\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3 = |\mathbf{v}||\mathbf{w}| \cos\theta$

## II.1 The scalar product

**Definition 3** *the scalar product*

The **scalar product** of two vectors is a number:

$$\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3 = |\mathbf{v}|\,|\mathbf{w}| \cos\theta,$$

where $\theta$ is the angle between $\mathbf{v}$ and $\mathbf{w}$, placed tail to tail. Both formulas are in the formula booklet.

**Example 4**

Find $\mathbf{v} \cdot \mathbf{w}$ for $\mathbf{v} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Multiply the components | $3 \times 1 + (-1) \times 4 + 2 \times (-2)$ |
| **Step 2** Add | $3 - 4 - 4 = -5$ |

**APPLY**

**1** $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 7 \\ 3 \end{pmatrix}$.

**2** $\mathbf{i} \cdot \mathbf{j}$.

## II.2 Angles and perpendicular vectors

**Property 2** *the angle, and perpendicular vectors*

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}.$$

Two non-zero vectors are **perpendicular** if and only if $\mathbf{v} \cdot \mathbf{w} = 0$. The sign of $\mathbf{v} \cdot \mathbf{w}$ gives the kind of angle: positive, acute; negative, obtuse.

**Example 5**

Find the angle between $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$.

| WHAT YOU DO | WHAT YOU WRITE |
|---|---|
| **Step 1** Scalar product | $2 - 2 - 4 = -4$ |
| **Step 2** Magnitudes | $\lvert\mathbf{v}\rvert = 3,\ \lvert\mathbf{w}\rvert = 3$ |
| **Step 3** $\cos\theta$ | $\cos\theta = \dfrac{-4}{3 \times 3} = -\dfrac{4}{9}$: obtuse |
| **Step 4** The angle (GDC) | $\theta = 116.4°\ \ (116.387\ldots)$ |

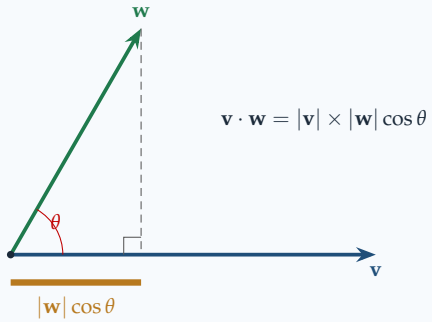



*Figure 2 — $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| \times$ the shadow of $\mathbf{w}$ on $\mathbf{v}$: positive while the angle is acute, zero at a right angle, negative beyond.*

**Watch out 2**

The angle at $B$ in a triangle $ABC$ uses $\overrightarrow{BA}$ and $\overrightarrow{BC}$, the two vectors **leaving** $B$. With $\overrightarrow{AB}$ and $\overrightarrow{BC}$ you find 180° minus the angle.

**APPLY**

**3** $\begin{pmatrix} t \\ 1 \\ 2 \end{pmatrix}$ perpendicular to $\begin{pmatrix} t \\ -3 \\ 1 \end{pmatrix}$: $t$.

**4** $A(3, 2, 1)$, $B(1, 0, 0)$, $C(1, 3, 4)$: the angle $A\hat{B}C$.

# III EXAM-STYLE QUESTIONS

from the shortest to the longest · every one in steps, with its marks

## A Vectors between points

**EXAM-STYLE QUESTION 1** Paper 1 · no calculator · [Maximum mark: 6]

The points $A$, $B$ and $C$ have coordinates $A(2, -1, 3)$, $B(5, 3, 3)$ and $C(1, 4, -2)$.

(a) Find $\overrightarrow{AB}$ and $\left|\overrightarrow{AB}\right|$. [2]

(b) Find a unit vector in the direction of $\overrightarrow{AB}$. [2]

(c) $ABCD$ is a parallelogram. Find the coordinates of $D$. [2]

**WORKING (a)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** End minus start | $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ | A1 |
| **Step 2** Magnitude | $\left\|\overrightarrow{AB}\right\| = \sqrt{9+16+0} = 5$ | A1 |

**WORKING (b)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Divide by the magnitude | $\frac{1}{5}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ | M1 |
| | *Divide by the magnitude.* | |
| **Step 2** The unit vector | $\begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix}$ | A1 |

**WORKING (c)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** $\overrightarrow{AD} = \overrightarrow{BC}$ | $\overrightarrow{AD} = \overrightarrow{BC} = \begin{pmatrix} -4 \\ 1 \\ -5 \end{pmatrix}$ | M1 |
| **Step 2** The coordinates of $D$ | $D(-2, 0, -2)$ | A1 |
| | $\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}$. | |

## B Perpendicular, then an angle

**EXAM-STYLE QUESTION 2** Paper 1 · no calculator · [Maximum mark: 6]

Let $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ k \\ 4 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$, where $k \in \mathbb{R}$.

(a) Find the value of $k$ for which $\mathbf{a}$ and $\mathbf{b}$ are perpendicular. [2]

(b) Find the cosine of the angle between $\mathbf{a}$ and $\mathbf{c}$. [3]

(c) Hence state whether this angle is acute or obtuse. [1]

**WORKING (a)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Scalar product $= 0$ | $\mathbf{a} \cdot \mathbf{b} = 2 + k - 8 = 0$<br>*Set the scalar product to zero.* | M1 |
| **Step 2** Solve | $k = 6$ | A1 |

**WORKING (b)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Scalar product | $\mathbf{a} \cdot \mathbf{c} = 6 + 0 - 8 = -2$ | A1 |
| **Step 2** Magnitudes | $\lvert\mathbf{a}\rvert = 3,\ \lvert\mathbf{c}\rvert = 5$ | A1 |
| **Step 3** $\cos\theta$ | $\cos\theta = \dfrac{-2}{3 \times 5} = -\dfrac{2}{15}$ | A1 |

**WORKING (c)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** The sign of the cosine | obtuse, because $\cos\theta < 0$<br>*The reason must be given.* | R1 |

## C A triangle on the GDC (Paper 2)

**EXAM-STYLE QUESTION 3** Paper 2 · calculator · [Maximum mark: 6]

The points $P(1, 0, 2)$, $Q(4, 1, -1)$ and $R(2, 5, 3)$ are the vertices of a triangle.

(a) Find $\overrightarrow{QP}$ and $\overrightarrow{QR}$. [2]

(b) Find the size of the angle $P\hat{Q}R$, in degrees. [2]

(c) Find the area of the triangle $PQR$. [2]

**WORKING (a)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** Vectors leaving $Q$ | $\overrightarrow{QP} = \begin{pmatrix} -3 \\ -1 \\ 3 \end{pmatrix}$ | A1 |
| **Step 2** The second vector | $\overrightarrow{QR} = \begin{pmatrix} -2 \\ 4 \\ 4 \end{pmatrix}$ | A1 |

**WORKING (b)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** $\cos\theta$ | $\cos\theta = \dfrac{6 - 4 + 12}{\sqrt{19} \times 6} = \dfrac{14}{6\sqrt{19}}$ | M1 |
| **Step 2** The angle | $P\hat{Q}R = 57.6°$ $(57.6355\ldots)$ | A1 |

**WORKING (c)**

| WHAT YOU DO | WHAT YOU WRITE | |
|---|---|---|
| **Step 1** $\frac{1}{2}ab\sin C$ | $\dfrac{1}{2} \times \sqrt{19} \times 6 \times \sin 57.6\ldots°$ | M1 |
| **Step 2** The area | $11.0$ $(11.0453\ldots)$ | A1 |

# IV WHICH METHOD? READ THE QUESTION

3.13 · the whole topic on one page

| THE QUESTION SAYS | YOU KNOW | YOU WRITE |
|---|---|---|
| *the coordinates of A and B* | $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ | the end minus the start D1 |
| *"the distance AB"* | $\left\lvert\overrightarrow{AB}\right\rvert$ | $\sqrt{v_1^2 + v_2^2 + v_3^2}$ D2 |
| *"a unit vector in the direction of"* | divide by the magnitude | $\frac{1}{\lvert\mathbf{v}\rvert}\mathbf{v}$ D2 |
| *a parallelogram ABCD, find D* | $\overrightarrow{AD} = \overrightarrow{BC}$ | $\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}$ P1 |
| *"show that A, B, C are collinear"* | $\overrightarrow{AC} = k\overrightarrow{AB}$ | find $k$, then say they share $A$ P1 |
| *"perpendicular"* | $\mathbf{v} \cdot \mathbf{w} = 0$ | an equation in the unknown P2 |
| *"the angle between"* | $\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\lvert\mathbf{v}\rvert\lvert\mathbf{w}\rvert}$ | scalar product, both magnitudes, then arccos P2 |
| *the angle ABC of a triangle* | the vectors leaving $B$ | $\overrightarrow{BA}$ and $\overrightarrow{BC}$ W2 |

**In the exam 1** *command terms*

**Find**: one vector as a column, or coordinates as a point. **Show that**: every step written, ending on the given result. **Hence**: use the result just found; a reason is expected for "acute or obtuse".

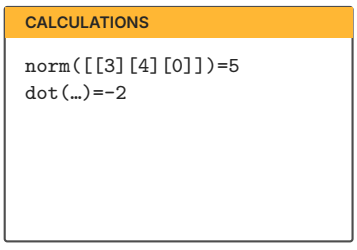
# V ON THE GDC: PAPER 2

AHL 3.12 · 3.13 · NumWorks · every calculator move of this topic

## 1 Magnitude and scalar product

Calculation → norm([[3][4][0]]) → dot([[2][1][-2]],[[3][0][4]])

Enter vectors as column matrices; `norm` gives the magnitude and `dot` the scalar product. **On paper:** the vector and its magnitude, as in the markscheme.



## 2 An angle in degrees

Settings → Angle measure: Degrees → acos(dot(u,v)/(norm(u)norm(v)))

Store the two vectors first, then one line gives the angle. Check the angle unit before you start. **On paper:** $\cos\theta$ with its numbers, then $\theta$ to three significant figures.

**CALCULATIONS**

```
acos(14/(√19×6))
=57.63551
```

### ANSWERS TO THE APPLY QUESTIONS

**I.1** $\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$

**I.2** $\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$

**I.3** $9$; $\frac{1}{9}\begin{pmatrix} 1 \\ -4 \\ 8 \end{pmatrix}$

**I.4** $3$

**I.5** $\mathbf{s} = \mathbf{p} + \mathbf{r} - \mathbf{q}$: $S(3, 5, 4)$

**I.6** factor $-2$: $k = -6$

**II.1** $7$

**II.2** $0$

**II.3** $t^2 - 1 = 0$: $t = \pm 1$

**II.4** $\overrightarrow{BA} \cdot \overrightarrow{BC} = 10$, $\cos = \frac{10}{3\times 5}$: $48.2°$

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