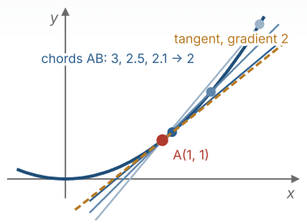
Differential calculus

rate → rules → tangent → shape → optimise → motion

I Limits and the derivative

AA 5.1

The gradient of the chord from $A(a, f(a))$ to a nearby point tends to the gradient of the **tangent** at A: that limit is the **derivative**.



$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

gradient of the curve = rate of change of y with respect to x

$f'(x) > 0$: f **increasing** · $f'(x) < 0$: f **decreasing** · $f'(a)$ has the **units** of y per unit of x.

Read a rate $V(t)$ litres after t minutes, $V'(10) = -4$: at t = 10 the volume is **decreasing** at 4 litres per minute.

Estimate a limit table of $\frac{f(1+h) - f(1)}{h}$ for h = 0.1, 0.01, 0.001: the values approach the gradient at x = 1.

Watch out $f'(a) = 0$ does not mean $f(a) = 0$ · "rate of change" always comes with its sign and its units.

II Rules of differentiation

AA 5.3 · 5.6

x^n	nx^{n-1}	e^x	e^x	$\sin x$	$\cos x$
$\ln x$	$\frac{1}{x}$	e^{kx}	ke^{kx}	$\cos x$	$-\sin x$

Chain $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ · **Product** $(uv)' = u'v + uv'$ · **Quotient** $(u/v)' = \frac{u'v - uv'}{v^2}$

Chain $(2x-1)^5 \to 5(2x-1)^4 \times 2 = 10(2x-1)^4 \cdot \cos 4x \to -4 \sin 4x \cdot \ln(x^2+1) \to \frac{2x}{x^2+1}$

Product $x^2e^{3x} \to 2xe^{3x} + 3x^2e^{3x} = xe^{3x}(2+3x)$ — factorise: the zeros are then visible.

Quotient $\frac{\sin x}{x} \to \frac{x \cos x - \sin x}{x^2}$

Rewrite first: $\sqrt{x} = x^{1/2}$, $\frac{3}{x^2} = 3x^{-2}$, split $\frac{x^2+4}{x} = x + 4x^{-1}$ before differentiating.

Watch out $(uv)' \neq u'v'$ · the chain factor: e^{3x} gives $3e^{3x}$, not $e^{3x} \cdot \ln(2x)$ gives $\frac{1}{x}$.

Unlocks a composite (a bracket to a power, e···, ln(...), sin(...)) → chain; two functions multiplied → product.

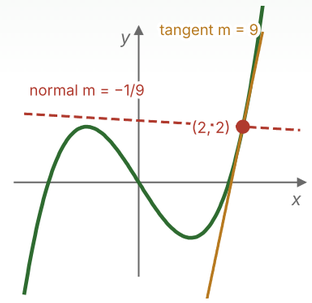
III Tangents and normals

AA 5.1 · 5.4

tangent at $x = a$: $y - f(a) = f'(a)(x - a)$

normal: gradient $-1/f'(a)$ — perpendicular to the tangent

1. the **point**: $f(a)$
2. the **gradient**: $f'(a)$
3. the line: $y - y_1 = m(x - x_1)$



Tangent to $y = x^3 - 3x$ at x = 2: point (2, 2), $f'(2) = 3(4) - 3 = 9$, so $y = 9x - 16$; normal $y - 2 = -\frac{1}{9}(x - 2)$.

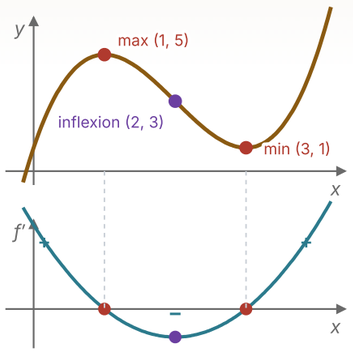
Gradient given where is the gradient 9? $3x^2 - 3 = 9$ gives $x = \pm 2$: **two** points, (2, 2) and (−2, −2).

Meets the curve again $x^3 - 3x = 9x - 16 \Leftrightarrow (x-2)^2(x+4) = 0$: at (−4, −52).

Watch out the gradient is $f'(a)$, a **number** — never put $f'(x)$ in the equation of the line · the normal is $-1/9$, not -9.

IV Stationary points, f′ and f″

AA 5.2 · 5.7 · 5.8



stationary: $f'(x) = 0$

$f''(a) < 0$: max · $f''(a) > 0$: min · $f''(a) = 0$: test the sign of f'

inflexion: $f''(x) = 0$ **and** f'' changes sign

$f'' > 0$ concave up · $f'' < 0$ concave down

Find and classify $f(x) = x^3 - 6x^2 + 9x + 1$: $f'(x) = 3(x-1)(x-3)$; $f''(1) = -6$: max (1, 5); $f''(3) = 6$: min (3, 1); $f''(x) = 6x - 12$: inflexion (2, 3).

From the graph of f′ to f: a zero of f′ where it crosses from + to − is a max of f; a turning point of f′ is an inflexion of f; f′ above the axis means f increasing.

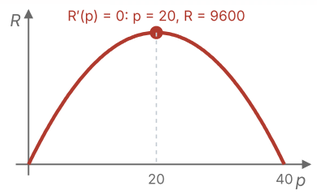
Watch out $f''(a) = 0$ alone proves nothing: $y = x^4$ has $f''(0) = 0$ and a minimum · state the **coordinates**, not just x.

V Optimisation

AA 5.8

1. one **variable**: use the constraint to remove the other
2. differentiate, solve $R' = 0$
3. **justify** the maximum: $R'' < 0$ or a sign table
4. answer the **question asked**, with units

Ticket price at p dollars, $N = 960 - 24p$ visitors. Revenue $R = p(960 - 24p) = 960p - 24p^2$; $R' = 960 - 48p = 0$, $p = 20$; $R'' = -48 < 0$: maximum, $R = 9600$ dollars, 480 visitors.



The domain matters: a length, a price or a time cannot be negative; if $R' = 0$ has no solution in the domain, the optimum is at an **endpoint**.

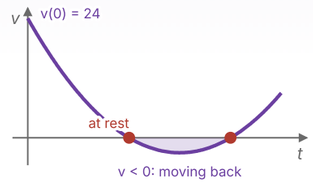
Watch out "show that the area is ..." gives you the function: use it, do not rebuild it · the question may ask for the **price**, the **revenue** or the **number of visitors**: three different answers.

VI Kinematics

AA 5.9

$s \to v = \frac{ds}{dt} \to a = \frac{dv}{dt}$

*at rest: $v = 0$ · speed = $|v|$ · speeding up: v and a **same sign***



At rest $s(t) = t^3 - 9t^2 + 24t$: $v = 3(t-2)(t-4)$, at rest at t = 2 and t = 4; $a = 6t - 18$.

Distance for 0 ≤ t ≤ 5: s(0) = 0, s(2) = 20, s(4) = 16, s(5) = 20 → the particle turns back twice: 20 + 4 + 4 = 28 m; the **displacement** is only 20 m. With a GDC: $\int_0^5 |v| \, dt = 28$.

Slowing down v and a of opposite signs: $0 < t < 2$ and $3 < t < 4$.

Watch out distance ≠ displacement as soon as v changes sign · "initially" means t = 0.

Command terms

Write down no working needed · **Find** the working that leads to it · **Show that** every step, the given result last · **Hence** use the previous part · **Justify** the nature: f″ or a sign table, in words.

Paper 1 · Paper 2

P1, no GDC: differentiate by hand, exact tangents, stationary points by factorising, f″ test.
P2, GDC: maximum of a model, solve f′(x) = k graphically, distance travelled as $\int|v|\,dt$, values to 3 s.f.

Booklet or memory?

booklet derivatives of x^n, sin, cos, e^x, ln x · chain, product, quotient rules · v = ds/dt, a = dv/dt
by heart how to **use** them: tangent in three steps, the sign of f′ and f″, "at rest" = v = 0, the optimisation recipe

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