Probability

count → combine → condition → branch → test independence → mix

I Sample spaces & probability

AA 4.5

Trial → **outcomes** → **sample space** U · an **event** is a subset of U. The probabilities of all outcomes add up to **1**.

equally likely: $\mathrm{P}(A) = \frac{n(A)}{n(U)}$

$\mathrm{P}(A') = 1 - \mathrm{P}(A)$

$0 \le P(A) \le 1$ · *"at least one" = 1 − P(none)*

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	**8**
3	4	5	6	7	**8**	9
4	5	6	7	**8**	9	10
5	6	7	**8**	9	10	11
6	7	**8**	9	10	11	12

sum 8: 5 cells → P = 5/36

Relative frequency k/N **estimates** $\mathrm{P}(A)$ · **expected number** $n\,\mathrm{P}(A)$.

Missing probability: a spinner has $\mathrm{P}(1) = 0.1$, $\mathrm{P}(2) = 0.3$, $\mathrm{P}(3) = k$, $\mathrm{P}(4) = 0.2$: the total is 1, so $k = 0.4$. **Estimate**: a pin lands point up 63 times in 150 throws, $\mathrm{P}(\text{up}) \approx 63/150 = 0.42$; in 500 throws, expect about $500 \times 0.42 = 210$.

Watch out two dice give **ordered** pairs: (2, 6) ≠ (6, 2) · the sums 2…12 are **not** equally likely · an expected number may be a decimal: $50 \times 0.03 = 1.5$.

II Venn diagrams & tables

AA 4.6

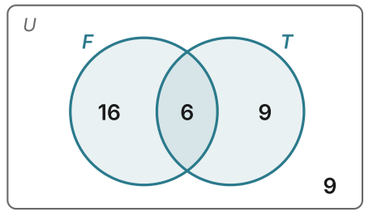
$A \cap B$ "and": in both · $A \cup B$ "or": in at least one, **both included**. **Mutually exclusive**: $\mathrm{P}(A \cap B) = 0$.

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$P(A \cap B') = P(A) - P(A \cap B)$ · $P(A' \cap B') = 1 - P(A \cup B)$

Filling a Venn diagram

1. the **intersection first**, in the middle
2. each "only" region by **subtraction**
3. outside: total minus inside
4. **check**: the regions add up to $n(U)$ or 1



40 students: 22 football, 15 tennis, 6 both

From the diagram: P(exactly one sport) = (16 + 9)/40 = 5/8. **Addition rule**: $\mathrm{P}(A) = 0.5$, $\mathrm{P}(B) = 0.4$, $\mathrm{P}(A \cup B) = 0.7$ give $0.7 = 0.5 + 0.4 - \mathrm{P}(A \cap B)$, so $\mathrm{P}(A \cap B) = 0.2$. **Two-way table** (50 students, 20 girls, 23 by bus, 5 girls by bus): $\mathrm{P}(\text{girl} \cup \text{bus}) = (20 + 23 - 5)/50 = 19/25$.

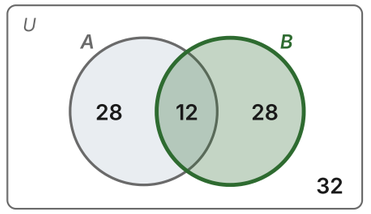
Watch out $n(A)$ is the **whole** circle, not "A only" · "or" includes both, "exactly one" does not.

III Conditional probability

AA 4.6 · 4.11

$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

equally likely: $n(A \cap B)/n(B)$ *— **B becomes the sample space***



Given B, only its 40 outcomes remain: $\mathrm{P}(A \mid B) = 12/40 = 0.3$, whereas $\mathrm{P}(A) = 0.4$.

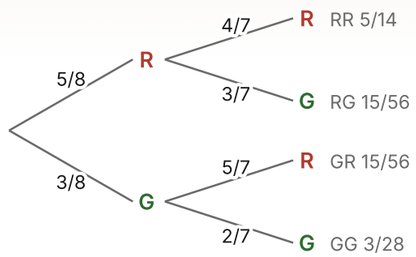
$\mathrm{P}(A \cap B) = \mathrm{P}(B)\,\mathrm{P}(A \mid B)$ · $\mathrm{P}(A' \mid B) = 1 - \mathrm{P}(A \mid B)$

Survey of II (22 football, 15 tennis, 6 both): $\mathrm{P}(T \mid F) = 6/22$ — **divide by the condition**, $n(F)$. With $\mathrm{P}(A) = 0.5$, $\mathrm{P}(B) = 0.4$, $\mathrm{P}(A \cap B) = 0.3$: $\mathrm{P}(A \mid B') = \frac{\mathrm{P}(A \cap B')}{\mathrm{P}(B')} = \frac{0.5 - 0.3}{1 - 0.4} = \frac{1}{3}$.

Watch out $\mathrm{P}(A \mid B) \neq \mathrm{P}(B \mid A)$: divide by the event **after "given"** · for $\mathrm{P}(A \mid B')$, draw the Venn diagram first.

IV Tree diagrams

AA 4.6



Second level = $\mathrm{P}(B \mid A)$. **Without replacement**: one fewer in total, one fewer of that kind.

Reading a tree

- branches from one point add to **1**
- **multiply** along a path
- **add** the paths of an event

Bag of 5 red, 3 green, two counters taken. **With** replacement: $\mathrm{P}(\text{one of each}) = \frac{5}{8} \times \frac{3}{8} + \frac{3}{8} \times \frac{5}{8} = 15/32$ — **two** paths. **Without** (the tree): $\mathrm{P}(\text{RR}) = \frac{5}{8} \times 4/7 = 5/14$.

Given the second stage

It rains with probability 0.3. Ana is late (L) with probability 0.4 if it rains, 0.1 if not. **Given that she is late, find P(rain).**

$\mathrm{P}(\text{rain} \mid L) = \frac{\mathrm{P}(\text{rain} \cap L)}{\mathrm{P}(L)} = \frac{0.3 \times 0.4}{0.3 \times 0.4 + 0.7 \times 0.1} = \frac{12}{19}$

*numerator: the path rain → L · denominator: **all** the paths that end with L*

Watch out "at least one" = $1 - \mathrm{P}(\text{none})$: one path instead of three.

V Independent events

AA 4.6 · 4.11

A and B independent $\Leftrightarrow$ $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B)$

$\Leftrightarrow P(A \mid B) = P(A) \Leftrightarrow P(A \mid B) = P(A \mid B')$ · *then A, B′ (and A′, B′) too*

Test it ($\mathrm{P}(A) = 0.5$, $\mathrm{P}(B) = 0.6$, $\mathrm{P}(A \cap B) = 0.4$): $\mathrm{P}(A)\,\mathrm{P}(B) = 0.3 \neq 0.4$ → not independent — **both numbers and a sentence**.

Use it (independent, $\mathrm{P}(A) = 0.3$, $\mathrm{P}(B) = 0.6$): $\mathrm{P}(A \cap B) = 0.3 \times 0.6 = 0.18$, then $\mathrm{P}(A \cup B) = 0.3 + 0.6 - 0.18 = 0.72$.

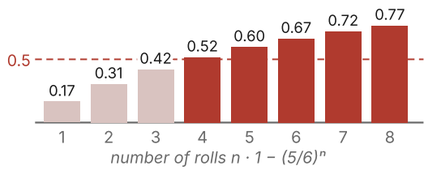
Repeated trials

multiply:

$\mathrm{P}(\text{no six in } n \text{ rolls}) = (5/6)^n$

At least one six in 4 rolls:

$1 - (5/6)^4 \approx 0.518$: **4 rolls** make a six more likely than not.



number of rolls n · 1 − (5/6)ⁿ

Watch out mutually exclusive is **not** independent: $\mathrm{P}(A \cap B) = 0$ but $\mathrm{P}(A)\mathrm{P}(B) > 0$ · a picture proves nothing.

VI Mixed questions — the whole chapter

Section B · Paper 1 & 2

One stem, five or six parts, each a different tool. Recognise the **trigger words**:

"list", two dice	sample space diagram
survey numbers, "only", "neither"	Venn from the middle
"given that"	÷ P(the condition)
"one after the other"	tree: × along, + paths
tree in one order, question in the other	reverse conditional
"determine whether independent"	compare two numbers
"least number of trials"	$1 - q^n > c$, table/solver

The medical test: 1.5 % ill, test positive 0.96 if ill, 0.08 if not. $\mathrm{P}(+) = 0.015 \times 0.96 + 0.985 \times 0.08 = 0.0932$ · $\mathrm{P}(\text{ill} \mid +) = 0.0144/0.0932 \approx 0.155$ — **comment**: most positives are healthy · 5000 tests → 466 positives expected.

Command terms

Write down read it off the diagram · **Show that** every step, the given value at the end · **Determine whether** numbers + verdict + sentence · **Comment** a sentence in context · **Hence or otherwise** the previous part is the shortcut.

Paper 1 · Paper 2

P1, no GDC: exact fractions — Venn, trees, conditional, a quadratic in p (reject the root outside [0, 1] **with the reason**).
P2, GDC: percentages, decimals to 3 s.f., "least number of trials" with a table or solver.

Booklet or memory?

booklet $n(A)/n(U)$ · complement · addition rule · mutually exclusive · $\mathrm{P}(A \mid B)$ · independent
by heart how to **use** them: "given" in context, multiply along / add paths, $1 - \mathrm{P}(\text{none})$

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