Transformations and rational functions

translate & reflect → stretch → in order → 1/x & asymptotes → rational functions → invert & combine

I Translations & reflections

AA 2.11

$y = f(x) + b$	translation b **up**
$y = f(x - a)$	translation a **right**
$y = f(x - a) + b$	translation by vector $(a; b)$
$y = -f(x)$	reflection in the ***x*-axis**
$y = f(-x)$	reflection in the ***y*-axis**

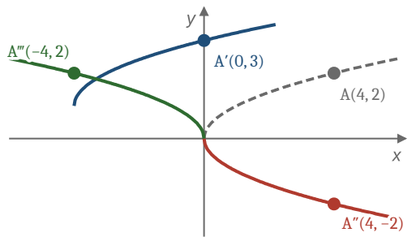


Image of a point

$(s, t) \to (s + a, t + b) \cdot (s, -t) \cdot (-s, t)$

E.g. $(3, -1)$ lies on $y = f(x)$; on $y = f(x - 2) + 5$ it becomes $(3 + 2, -1 + 5) = (5, 4)$.

Domain & range

transform the **ends**; a reflection **swaps** them.

Describe $y = x^2 \to y = (x + 4)^2$: write $x + 4 = x - (-4)$, so $a = -4$: translation 4 units **left**.
Find the equation when $y = x^2$ is moved 1 right, 2 down: $g(x) = (x - 1)^2 - 2$, vertex $(1, -2)$.

Watch out inside the bracket the sign works **backwards**: $f(x + 2)$ is 2 **left** · "it moves" earns nothing: name it, give direction and size.

II Stretches

AA 2.11

$y = p\,f(x)$ vertical stretch, s.f. p · $y = f(qx)$ horizontal stretch, s.f. $\frac{1}{q}$

points on the x-axis (resp. y-axis) do not move

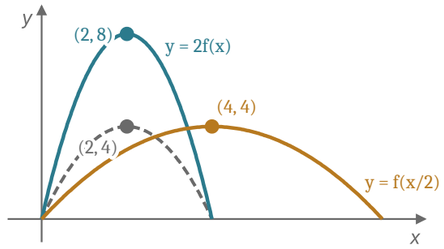


Image of a point

$(s, t) \to (s, pt)$ on $y = p\,f(x)$

$(s, t) \to (\frac{s}{q}, t)$ on $y = f(qx)$

The maximum $(2, 4)$ of $4x - x^2$ (dashed) goes to $(2, 8)$ and $(4, 4)$; the domain $[0, 4]$ becomes $[0, 8]$.

Describe $y = f(x) \to y = f(x/2)$: here $q = ½$, so a horizontal stretch with scale factor $1/q = 2$.

Watch out $y = f(2x)$ is a scale factor ½, not 2 · "shrink" is not an IB word: "stretch with scale factor ½".

II Sequences of transformations

AA 2.11

The order matters

stretch, **then** translate: $y = p\,f(x) + b \neq p(f(x) + b) = p\,f(x) + pb$

Describing $y = p\,f(x - a) + b$

1. **inside the bracket** — horizontal: translation by a, stretch s.f. $1/q$, reflection in the y-axis $(-x)$
2. **in front of *f*** — vertical stretch s.f. $|p|$, reflection in the x-axis if $p < 0$
3. **added at the end** — vertical translation by b, after the stretch
4. **check** one point: $(s, t) \to (s + a, pt + b)$

Describe $y = x^2 \to y = -2x^2 + 3$: vertical stretch s.f. 2 and reflection in the x-axis, **then** 3 up · check $(2, 4) \to (2, -2 \times 4 + 3) = (2, -5)$.

$(2, 5)$ lies on $y = f(x)$. **Image on** $y = 2f(3x) + 4$: $x = 2 \div 3 = ⅔$ (inside: divide by 3), $y = 2 \times 5 + 4 = 14 \to (⅔, 14)$.

Watch out $2f(x) + 4 \neq 2(f(x) + 4)$: stretch first, then translate. Give **every** transformation its direction and size.

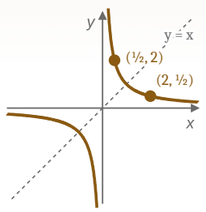
III Reciprocal function & asymptotes

AA 2.8

$y = \frac{k}{x}$, $x \neq 0$

*asymptotes $x = 0$, $y = 0$ · domain $x \neq 0$ · range $y \neq 0$ · **self-inverse**: symmetric in $y = x$*

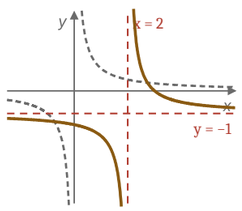
Asymptote : a **line** the curve gets as close to as we want — given by its **equation**.



Translated reciprocal

$y = \frac{k}{x - a} + b$

asymptotes $x = a$, $y = b$ · domain $x \neq a$ · range $y \neq b$



$\frac{1}{x - 2} - 1$: $\frac{1}{x}$ translated by $(2; -1)$

Asymptotes of $y = \frac{3}{x + 1} - 2$: $x + 1 = x - (-1)$, so $a = -1$, $b = -2$: $x = -1$, $y = -2$.

Describe $y = \frac{1}{x} \to y = -\frac{5}{x}$: stretch s.f. 5, reflection in x-axis.

Watch out $y = \frac{1}{x + 3}$: asymptote $x = -3$, not 3 · answer "$x = 2$", not "2".

IV Rational functions: features & sketch

AA 2.8

$f(x) = \frac{ax + b}{cx + d}$, $c \neq 0$ · domain $x \neq -\frac{d}{c}$

Asymptotes

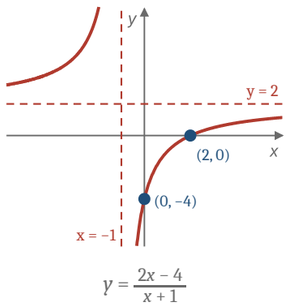
$x = -\frac{d}{c}$ (denominator zero) · $y = \frac{a}{c}$ (ratio of the x-terms) · range $y \neq \frac{a}{c}$

Intercepts

$(0, \frac{b}{d}) = f(0)$ · $(-\frac{b}{a}, 0)$: numerator zero

Sketch

1. asymptotes dashed, **labelled with equations**
2. intercepts **with coordinates**
3. the branch through them
4. the other branch, **opposite corner**



$y = \frac{2x - 4}{x + 1}$

Parameters from the asymptotes: $f(x) = \frac{ax + 3}{x + c}$ has asymptotes $x = 2$ and $y = 4$.
Denominator zero at 2: $2 + c = 0$, $c = -2$; ratio of the x-terms: $a/1 = 4$, $a = 4$.

Watch out $y = \frac{a}{c}$, not $\frac{b}{d}$ (that is the y-intercept): $\frac{3x + 4}{2x - 5}$ gives $y = \frac{3}{2}$, $x = \frac{5}{2}$.

IV Inverse, other forms — and the exam

AA 2.8 · 2.11

$f^{-1}(x) = \frac{dx - b}{a - cx}$, $x \neq \frac{a}{c}$

*the asymptotes **swap** · **self-inverse** ⇔ $d = -a$ · or: swap x and y, make y the subject*

Inverse of $f(x) = \frac{2x + 1}{x - 3}$: $(a, b, c, d) = (2, 1, 1, -3)$, so $f^{-1}(x) = \frac{-3x - 1}{2 - x} = \frac{3x + 1}{x - 2}$.

The form $a + \frac{k}{x - c}$

$\frac{2x + 5}{x + 1} = \frac{2(x + 1) + 3}{x + 1} = 2 + \frac{3}{x + 1}$ → from $\frac{1}{x}$: vertical stretch s.f. 3, then 1 left and 2 up. The bridge between parts **II, III and IV**.

The mixed questions (part V)

- **Show that** $f(x) = \frac{ax + b}{cx + d}$ → asymptotes → intercepts → **hence** the transformations from $\frac{1}{x}$.
- $g(x) = f(x + 1) - 1$: simplify, then read the asymptotes **moved by the translation**.
- A **model** $F(t) = \frac{800t + 200}{t + 4}$: initial value $F(0)$, solve $F(t) = 600$, the asymptote is the **limit never reached**.

GDC Paper 2 checks: Find ▸ Zeros, Maximum, Point of intersection — exact values on Paper 1.

Command terms

Describe fully the name + direction + size, for every transformation · **Write down** answer alone (asymptotes, images of points) · **Sketch** asymptotes dashed with equations, intercepts with coordinates, both branches · **Hence** use the form just shown.

Paper 1 · Paper 2

P1, no GDC: describe transformations, images of points, asymptotes and intercepts, f^{-1}, parameters — exact values ($-4/3$, not -1.33).
P2, GDC: models in context, intersections, zeros and maxima to check a transformation. 3 s.f.

Booklet or memory?

by heart nothing here is in the booklet: the six transformations · inside / in front / at the end · $x = -d/c$, $y = a/c$ · f^{-1} and $d = -a$
GDC only checking a sketch, a zero, a maximum

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