Differential calculus — markscheme

Revision sheet "every question type", Analysis and approaches SL · 9 questions · 109 marks

Question 1 Type 1 — differentiate

(a) $f'(x) = 4(3x-2)^3 \times 3 = 12(3x-2)^3$ A1 $f'(1) = 12$ A1

The factor 3 of the chain rule must appear for the first A1.

(b) $g'(x) = 1 \times \mathrm{e}^{-2x} + x \times (-2\mathrm{e}^{-2x})$ M1A1A1
$= \mathrm{e}^{-2x}(1-2x)$ AG

(c) $\mathrm{e}^{-2x} > 0$, so $1-2x = 0$, $x = \frac{1}{2}$ M1 $\left(\frac{1}{2}, \frac{1}{2\mathrm{e}}\right)$ A1

(d) $h'(x) = \dfrac{\frac{1}{x} \times x - \ln x \times 1}{x^2} = \dfrac{1-\ln x}{x^2}$ M1A1A1

(e) $h'(\mathrm{e}) = \dfrac{1-\ln \mathrm{e}}{\mathrm{e}^2} = 0$, so the tangent at $x = \mathrm{e}$ is horizontal R1

Question 2 Type 2 — tangent and normal

(a) $f(x) = x^2 + 8x^{-1}$, $f'(x) = 2x - 8x^{-2} = 2x - \dfrac{8}{x^2}$ M1A1

(b) $f(2) = 4+4 = 8$ A1 $f'(2) = 4-2 = 2$ A1 $y - 8 = 2(x-2)$ M1 $y = 2x+4$ AG

(c) gradient of the normal $-\frac{1}{2}$: $y - 8 = -\frac{1}{2}(x-2)$, $y = -\frac{1}{2}x + 9$ M1A1

(d) $A(-2,0)$, $B(18,0)$ A1A1 area $= \frac{1}{2} \times 20 \times 8 = 80$ A1

(e) $2x - \dfrac{8}{x^2} = 0$, $x^3 = 4$, $x = \sqrt[3]{4}$ (≈ 1.59) M1A1

Question 3 Type 3 — stationary points, nature, inflexion

(a) $f'(x) = 6x^2 + 6x - 12 = 6(x+2)(x-1)$ A1A1

(b) $f'(x) = 0$: $x = -2$ or $x = 1$ M1A1 $(-2,\ 24)$ and $(1,\ -3)$ A1A1

(c) $f''(x) = 12x + 6$ A1 $f''(-2) = -18 < 0$: maximum; $f''(1) = 18 > 0$: minimum R1R1

(d) $-2 < x < 1$ A2

(e) $f''(x) = 0$: $x = -\frac{1}{2}$, and f'' changes sign there M1R1 $\left(-\frac{1}{2},\ \frac{21}{2}\right)$ A1

(f) three solutions when the line $y = k$ lies strictly between the turning values: $-3 < k < 24$ A1A1

A1 for the endpoints, A1 for the strict inequalities.

Question 4 Type 4 — from the graph of f' to f

(a) $x = -1$ A1 because f' changes from positive to negative there R1

(b) $-1 < x < 3$ A1

(c) $x = 1$ A1 because f' has a turning point there (f'' changes sign) R1

(d) $x > 1$ A1

(e) gradient $f'(0) = -3$ (read from the graph) A1 $y = -3x + 2$ A1

(f) $f'(x) < 0$ on $]-1, 3[$, so f is decreasing between them R1 $f(-1) > f(3)$ A1

Question 5 Type 5 — find the parameters

(a) $f'(x) = 3ax^2 + 2bx - 9$, $f''(x) = 6ax + 2b$ A1A1

(b) $f'(3) = 0$: $27a + 6b - 9 = 0$ M1A1 $9a + 2b = 3$ AG

(c) $f''(1) = 0$: $6a + 2b = 0$ M1 subtracting: $3a = 3$, $a = 1$, $b = -3$ A1A1

(d) $f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1)$ M1 other stationary point $(-1,\ 10)$ A1A1 $f''(-1) = -12 < 0$: maximum R1

(e) $f(0) = 5$, $f'(0) = -9$: $y = -9x + 5$ A1A1

Question 6 Type 6 — optimisation

(a) time $= \dfrac{240}{v}$ hours M1 $C = \dfrac{240}{v}\left(0.03v^2 + 120\right) = 7.2v + \dfrac{28\,800}{v}$ A1AG

(b) $C'(v) = 7.2 - \dfrac{28\,800}{v^2}$ A1A1

(c) $7.2 - \dfrac{28\,800}{v^2} = 0$ M1 $v^2 = 4000$ A1 $v \approx 63.2 \text{ km h}^{-1}$ A1

(d) $C''(v) = \dfrac{57\,600}{v^3} > 0$ for $v > 0$ M1 so a minimum R1

(e) $C \approx 910.74$ dollars (accept 911) A1

(f) $C'(v) < 0$ for $0 < v < 63.2$, so C is decreasing on $]0, 50]$ R1 the least cost is at $v = 50$ M1 $C(50) = 360 + 576 = 936$ dollars A1

Question 7 Type 7 — kinematics

(a) $v(0) = 2 \text{ m s}^{-1}$ A1

(b) $v(t) = 0$ (GDC): $t \approx 0.841$ s, $t \approx 5.47$ s A1A1

(c) $a(t) = 0.5\,\mathrm{e}^{0.5t} - 3$ A1A1

(d) $a(t) = 0$: $\mathrm{e}^{0.5t} = 6$, $t = 2\ln 6$ (≈ 3.58) M1A1 least velocity $v = 7 - 6\ln 6 \approx -3.75 \text{ m s}^{-1}$ A1

(e) $v(2) = \mathrm{e} - 5 \approx -2.28 < 0$ and $a(2) = 0.5\mathrm{e} - 3 \approx -1.64 < 0$ M1 same sign: speeding up R1

(f) distance $= \displaystyle\int_0^6 |v(t)|\,\mathrm{d}t$ M1 ≈ 13.0 m A1

Not $\int_0^6 v\,\mathrm{d}t$: the particle turns back twice.

Question 8 Type 8 — rates of change in a model

(a) $N(0) = \dfrac{5000}{25} = 200$ people A1

(b) $N'(3)$ (GDC) ≈ 482 people per day M1A1

(c) three days after it starts, about 482 more people hear the rumour each day R1

(d) fastest when $N''(t) = 0$ (or the maximum of N') M1 $t = \dfrac{\ln 24}{0.6} \approx 5.30$ days A1 $N = 2500$ people A1

(e) 5000 people A1

Question 9 Without a label — types 1, 3, 2 and 3

(a) *Type 1.* $f'(x) = 2x\,\mathrm{e}^x + (x^2 - 3)\,\mathrm{e}^x$ M1A1 $= (x^2 + 2x - 3)\,\mathrm{e}^x = (x+3)(x-1)\,\mathrm{e}^x$ A1AG

(b) *Type 3.* $\mathrm{e}^x > 0$, so $x = -3$ or $x = 1$ M1 $\left(-3,\ 6\mathrm{e}^{-3}\right)$ and $\left(1,\ -2\mathrm{e}\right)$ A1A1

(c) sign of f': $+, -, +$ around -3 and 1 M1 maximum at $x = -3$, minimum at $x = 1$ A1

(d) *Type 2.* $f(0) = -3$, $f'(0) = -3$: $y = -3x - 3$ A1A1

(e) *Type 3, read on a sketch.* $f > 0$ for $x < -\sqrt{3}$, rising from 0 to the maximum $6\mathrm{e}^{-3}$, then down to $-2\mathrm{e}$, then up without bound M1 exactly two solutions when $k = 6\mathrm{e}^{-3}$ A1 or $-2\mathrm{e} < k \leq 0$ A1A1

A1 for $-2\mathrm{e} < k$, A1 for $k \leq 0$: at $k = 0$ the solutions are $\pm\sqrt{3}$ only, since f stays positive on the left.