Differential calculus — every question type

Rule of the game. Before the first line of working, *name the type* and its move: that is what Question 9 tests, without a label. Three slips cost marks every session: putting $f'(x)$ instead of the number $f'(a)$ in the equation of a tangent; classifying a stationary point with nothing written (no f'', no sign table, no mark); answering the variable when the question asks for the quantity (the speed, not the cost — or the reverse).

Type 1 · Differentiate

Type 1 — Differentiate: rules and a value
You recognise it by "find $f'(x)$", a bracket to a power, $e^{\cdots}$, $\ln(\dots)$, two functions multiplied or divided, "show that $f'(x) = \dots$".
The move rewrite powers first ($\sqrt{x} = x^{1/2}$, $\frac{1}{x^2} = x^{-2}$); chain for a composite, product for uv, quotient for $\frac{u}{v}$; factorise the result — the zeros of f' are then visible.

Worked example — the chain rule Paper 1 · 3 marks

Let $f(x) = (2x+5)^3$. Find $f'(x)$ and $f'(-2)$.

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Spot the inside and the outside	inside $2x+5$, outside $(\ )^3$	
Step 2 Differentiate the outside, keep the inside	$3(2x+5)^2$	M1
Step 3 Multiply by the derivative of the inside	$f'(x) = 3(2x+5)^2 \times 2 = 6(2x+5)^2$	A1
Step 4 Substitute the number	$f'(-2) = 6(1)^2 = 6$	A1

Question 1 ★★ Three functions, three rules — Paper 1 · 11 marks

Let $f(x) = (3x-2)^4$, $g(x) = x\,e^{-2x}$ and $h(x) = \dfrac{\ln x}{x}$ for $x > 0$.

(a) Find $f'(x)$ and $f'(1)$. [2]

(b) Show that $g'(x) = e^{-2x}(1-2x)$. [3]

(c) Hence find the exact coordinates of the point on the graph of g where the gradient is zero. [2]

(d) Find $h'(x)$. [3]

(e) Show that the graph of h has a horizontal tangent at $x = e$. [1]

Type 2 · Tangent and normal

Type 2 — Tangent and normal
You recognise it by "equation of the tangent / normal at the point where $x = a$", "the tangent meets the axis at", "the point where the gradient is m".
The move point $(a, f(a))$, gradient $f'(a)$ (normal: $-1/f'(a)$), then $y - y_1 = m(x - x_1)$; a given gradient means solve $f'(x) = m$.

Worked example — a tangent Paper 1 · 4 marks

Find the equation of the tangent to the curve $y = x^3 - 4x$ at the point where $x = 1$.

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Find the point	$y = 1 - 4 = -3$: $(1, -3)$	A1
Step 2 Differentiate	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 4$	A1
Step 3 Gradient: substitute $x = 1$	$m = 3 - 4 = -1$	M1
Step 4 Use $y - y_1 = m(x - x_1)$	$y + 3 = -(x - 1)$, $y = -x - 2$	A1

Question 2 ★★ A triangle made by two lines Paper 1 · 12 marks

Let $f(x) = x^2 + \dfrac{8}{x}$, $x \neq 0$. The point P on the graph of f has x-coordinate 2.

(a) Find $f'(x)$. [2]

(b) Show that the tangent to the graph of f at P has equation $y = 2x + 4$. [3]

(c) Find the equation of the normal at P, in the form $y = mx + c$. [2]

(d) The tangent and the normal at P meet the x-axis at A and B. Find the area of triangle APB. [3]

(e) Find the x-coordinate of the point where the tangent to the graph of f is horizontal. [2]

Type 3 · Stationary points, nature, inflexion

Type 3 — Stationary points, nature, inflexion

You recognise it by "find the coordinates of the stationary points", "determine their nature", "increasing / decreasing", "point of inflexion", "$f(x) = k$ has three solutions".

The move $f'(x) = 0$; nature by $f''(a)$ or a sign table of f'; inflexion where $f'' = 0$ *and* changes sign; for $f(x) = k$, read the sketch between the two turning values.

Worked example — stationary points and their nature Paper 1 · 6 marks

Find the coordinates of the stationary points of the curve $y = x^3 - 3x^2 + 1$, and determine their nature.

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Differentiate	$y' = 3x^2 - 6x$	A1
Step 2 Set $y' = 0$ and factorise	$3x(x - 2) = 0$: $x = 0$ or $x = 2$	M1 A1
Step 3 Find each y	$(0, 1)$ and $(2, -3)$	A1
Step 4 Classify with y''	$y'' = 6x - 6$: $y''(0) = -6 < 0$ max; $y''(2) = 6 > 0$ min	M1 A1

Question 3 ★★★ A cubic, read completely Paper 1 · 16 marks

Let $f(x) = 2x^3 + 3x^2 - 12x + 4$.

(a) Find $f'(x)$, giving your answer in factorised form. [2]

(b) Find the coordinates of the two stationary points of the graph of f. [4]

(c) Use $f''(x)$ to determine the nature of each stationary point. [3]

(d) Write down the interval on which f is decreasing. [2]

(e) Find the coordinates of the point of inflexion. [3]

(f) Find the set of values of k for which the equation $f(x) = k$ has three real solutions. [2]

Type 4 · From the graph of f' to f

Type 4 — From the graph of f' to f
You recognise it by the graph shown is the graph of f' (or of f''), not of f.
The move zeros of f' = stationary points of f (+ to −: max); sign of f' = increasing or decreasing; turning points of f' = inflexions of f; f' increasing = f concave up.

Worked example — reading the graph of f' Paper 1 · 4 marks

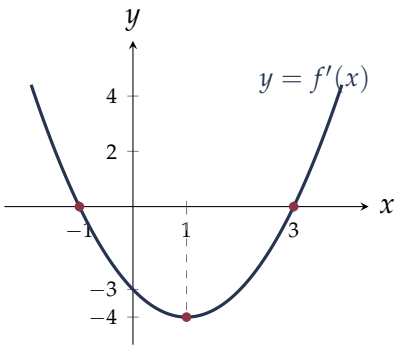
The graph of $y = f'(x)$ is the parabola $y = 4 - x^2$: it crosses the x-axis at $x = -2$ and $x = 2$, and has its highest point at $(0, 4)$. (a) Find the x-coordinate of the local minimum of f. (b) Find the x-coordinate of the point of inflexion of f. Justify each answer.

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Read the zeros of f'	$f' = 0$ at $x = -2$ and $x = 2$: stationary points of f	
Step 2 Read the sign of f' around each zero	(a) at $x = -2$, f' goes from $-$ to $+$: minimum at $x = -2$	R1 A1
Step 3 Find where f' turns	(b) f' has a maximum at $x = 0$: inflexion of f at $x = 0$	R1 A1

Question 4 ★★ The graph is the derivative

Paper 1 · 10 marks

The diagram shows the graph of $y = f'(x)$, the *derivative* of a function f. It crosses the x-axis at $x = -1$ and $x = 3$, and has a minimum point at $(1, -4)$.



(a) Find the x-coordinate of the local maximum of f. Justify your answer. [2]

(b) Write down the interval on which f is decreasing. [1]

(c) Find the x-coordinate of the point of inflexion of the graph of f. Justify your answer. [2]

(d) Write down the set of values of x for which the graph of f is concave up. [1]

(e) Given that $f(0) = 2$, find the equation of the tangent to the graph of f at $x = 0$. [2]

(f) State, with a reason, which is greater: $f(-1)$ or $f(3)$. [2]

Type 5 · Find the parameters

Type 5 — Find the parameters
You recognise it by a, b, k unknown and conditions: "a stationary point at $x = 3$", "an inflexion at $x = 1$", "the tangent at ... is ...".
The move one equation per condition — $f'(3) = 0$, $f''(1) = 0$, $f(0) = 5$, $f'(0) = m$ — then solve the system; check the conditions with the values found.

Worked example — two conditions, two unknowns Paper 1 · 5 marks

The curve $y = x^3 + ax^2 + bx$ has a stationary point at $(1, -4)$. Find a and b.

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Differentiate, letters kept	$y' = 3x^2 + 2ax + b$	A1
Step 2 One equation per condition	$y'(1) = 0$: $3 + 2a + b = 0$; $y(1) = -4$: $1 + a + b = -4$	M1 M1
Step 3 Solve the system	subtract: $2 + a = 4$, $a = 2$, $b = -7$	A1 A1

Question 5 ⋆⋆⋆ Two conditions, two unknowns Paper 1 · 13 marks

Let $f(x) = ax^3 + bx^2 - 9x + 5$, where a and b are constants.

(a) Find $f'(x)$ and $f''(x)$ in terms of a and b. [2]

(b) The graph of f has a stationary point at $x = 3$. Show that $9a + 2b = 3$. [2]

(c) The graph of f also has a point of inflexion at $x = 1$. Find the value of a and the value of b. [3]

(d) Find the coordinates of the other stationary point, and determine its nature. [4]

(e) Write down the equation of the tangent to the graph of f at the point where it meets the y-axis. [2]

Type 6 · Optimisation in context

Type 6 — Optimisation in context

You recognise it by "maximum", "minimum", "greatest", "least cost", "show that $C = \ldots$" then "find the value of v that ...".

The move one variable (use the given expression or the constraint); $C' = 0$; justify with C'' or a sign table; answer the quantity asked, with units; check the domain — the optimum may sit at an endpoint.

Worked example — the least fuel Paper 2 · 5 marks

A boat travelling at v km h^{-1} uses $F(v) = 0.02v + \dfrac{8}{v}$ litres of fuel per kilometre, for $v > 0$. Find the speed that uses the least fuel per kilometre, justify that it is a minimum, and find that least amount.

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Differentiate	$F'(v) = 0.02 - \dfrac{8}{v^2}$	A1
Step 2 Set $F' = 0$ and solve	$v^2 = 400$, $v = 20$ km h^{-1}	M1 A1
Step 3 Justify with F''	$F''(v) = \dfrac{16}{v^3} > 0$: minimum	R1
Step 4 Answer the quantity asked	$F(20) = 0.4 + 0.4 = 0.8$ litres per km	A1

Question 6 ⋆⋆⋆ The cheapest speed Paper 2 · 13 marks

A delivery lorry travels 240 km at a constant speed of v km h^{-1}. Running it costs $0.03v^2$ dollars per hour in fuel and 120 dollars per hour for the driver.

(a) Show that the total cost of the journey, in dollars, is $C(v) = 7.2v + \dfrac{28\,800}{v}$. [2]

(b) Find $C'(v)$. [2]

(c) Find the speed that minimises the cost of the journey. [3]

(d) Justify that this speed gives a minimum. [2]

(e) Find the minimum cost. [1]

(f) On one road the speed is limited to 50 km h^{-1}. Find the least cost of the journey on that road, justifying your answer. [3]

Type 7 · Kinematics

Type 7 — Kinematics
You recognise it by displacement s, velocity v, acceleration a, "at rest", "speeding up", "distance travelled".
The move $v = \frac{ds}{dt}$, $a = \frac{dv}{dt}$; at rest: $v = 0$; speeding up when v and a have the same sign; distance $= \int |v|\,dt$ (GDC), not the displacement.

Worked example — at rest Paper 1 · 4 marks

A particle moves so that its displacement, in metres, after t seconds is $s(t) = 2t^3 - 15t^2 + 36t$, $t \geq 0$. Find the times when it is at rest, and its acceleration at those times.

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Differentiate s to get v	$v = 6t^2 - 30t + 36$	A1
Step 2 At rest: $v = 0$	$6(t-2)(t-3) = 0$: $t = 2$ s, $t = 3$ s	M1 A1
Step 3 Differentiate v, substitute	$a = 12t - 30$: $a(2) = -6$, $a(3) = 6\ \text{m s}^{-2}$	A1

Question 7 ⋆⋆ A particle that turns back

Paper 2 · 12 marks

A particle moves along a straight line. Its velocity, in m s^{-1}, t seconds after it starts, is

$$v(t) = e^{0.5t} - 3t + 1, \qquad 0 \leq t \leq 6.$$

(a) Write down the initial velocity. [1]

(b) Find the values of t for which the particle is at rest. [2]

(c) Find an expression for the acceleration $a(t)$. [2]

(d) Find the exact value of t at which the velocity is least, and the least velocity. [3]

(e) Determine whether the particle is speeding up or slowing down at $t = 2$. Justify your answer. [2]

(f) Find the total distance travelled in the first 6 seconds. [2]

Type 8 · Rates of change in a model

Type 8 — Rates of change in a model
You recognise it by a model $N(t)$, $P(t)$, $T(t)$ and "rate of change", "increasing fastest", "interpret", "in the long run".
The move rate $= N'(t)$, with units and sign; fastest rate: $N''(t) = 0$; long run: the limit as $t \to \infty$; interpret in one sentence, in context.

Worked example — a rate in a model Paper 2 · 4 marks

The temperature of a drink, in °C, t minutes after it is poured, is $T(t) = 20 + 65\,e^{-0.08t}$. Find the rate of change of T at $t = 5$, interpret it, and write down the temperature in the long run.

WHAT YOU DO	WHAT YOU WRITE	
Step 1 Differentiate	$T'(t) = -5.2\,e^{-0.08t}$	A1
Step 2 Substitute $t = 5$ (GDC)	$T'(5) = -3.49$ $(-3.4857\ldots)$	A1
Step 3 Interpret: sign and units	the drink is cooling by about 3.49 °C per minute	R1
Step 4 Long run: $e^{-0.08t} \to 0$	20 °C	A1

Question 8 ⋆⋆⋆ The rumour — Paper 2 · 8 marks

The number of people who have heard a rumour, t days after it starts, is modelled by

$$N(t) = \frac{5000}{1 + 24\,\mathrm{e}^{-0.6t}}, \qquad t \geq 0.$$

(a) Find the number of people who know the rumour at the start. [1]

(b) Find the rate at which N is increasing when $t = 3$. [2]

(c) Interpret your answer to part (b) in context. [1]

(d) Find the value of t at which the rumour is spreading fastest, and the number of people who know it at that time. [3]

(e) Write down the number of people the model predicts will eventually hear the rumour. [1]

Without a label · name the types first

Question 9 ⋆⋆⋆ Four types in one question — Paper 1 · 14 marks

Let $f(x) = (x^2 - 3)\,\mathrm{e}^x$.

(a) Show that $f'(x) = (x + 3)(x - 1)\,\mathrm{e}^x$. [3]

(b) Find the exact coordinates of the stationary points of the graph of f. [3]

(c) Determine the nature of each stationary point. [2]

(d) Find the equation of the tangent to the graph of f at the point where $x = 0$. [2]

(e) Given that $f(x) \to 0$ as $x \to -\infty$, find the set of values of k for which the equation $f(x) = k$ has exactly two solutions. [4]

Before you start Question 9, write the type of each part in the margin. Answers, with the marks, are in the separate markscheme.